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C O E F F I C I E N T S I N B O R D I S M

- by -

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INTRODUCTION

In this paper is given a treatment of coefficients for a vast class of (co)-bordism theories of manifolds with singularities. If $h(-)$ is one of these theories and G is an abelian group we introduce coefficients G into h in such a way that

(a) $h(-;G)$ is a functor on the category of all abelian groups;

(b) the universal-coefficient sequence is natural on the category of all abelian groups. Consequently, by [5], it is pure and hence, by algebra, it splits for a vast class of abelian groups, including all groups of finite type.

(c) If G is an R-module, R any commutative ring with unit, $h(-;G)$ inherits an R-module and $h(\text{point};R)$-module structure in a natural way. It follows from Dold [3] that there is a generalized universal-coefficient sequence consisting of a spectral sequence running $\mathrm{Tor}_p(h^q(-;R),G) \Longrightarrow h(-;G)$.

(d) When $h(-) = H(-;Z)$ singular (co)homology with integer coefficients, the definition coincides with the usual one given by means of chain complexes.

(e) The method works to introduce local coefficients in the cohomology theory $h^*(-)$; i.e. if F is any sheaf of modules over a compact polyhedron X, then $h^*(X;F)$ is defined and it is a functor on sheaves.

(f) If F/X is a 'nice' sheaf (in the sense of III.3), there is a spectral sequence running

$$H^p(X;h^q_F) \Longrightarrow h^*(X;F) \qquad (*)$$

where h^q_F is the graded sheaf obtained from F by applying the functor $h^q(\text{point}; -)$ and H is singular cohomology. When F is constant, (*) reduces to the usual type. A comparison theorem is deduced from (*) by means of the Mapping Theorem between spectral sequences.

We restrict ourselves to the class of (co)bordism theories h(-) for which the cone over a framed circle is an allowable singularity. This assumption enables us to perform all the geometric constructions that we need. Many important cohomology theories turn out to be in the range that we are considering. For instance singular (co)homology; p.l., smooth, complex (co)bordism and others; a particularly interesting example is given by real K-theory at odd primes. In order to see it the essential tool is Sullivan-map $S: \Omega_*^{smooth}(-)_{odd} \longrightarrow KO_{*\ odd}^{(-)}$. The method used in this paper allows us to interpret $\Omega_*^{smooth}(-)_{odd}$ as a geometric homology theory and then a killing-kernel process due to Sullivan applies to represent $KO_{*\ odd}$ as a bordism theory (see §II 4 for details).

Here is a brief sketch of the approach used to define coefficients. If G is an abelian group and $\wp \xrightarrow{\varepsilon} G$ is a free resolution of length $\leqslant 4$ with some extra-structure, we construct a class of links, $\mathcal{L}$, associated to $\wp$. Then a $\wp$ - manifold consists of a trivialized stratified set whose links are in $\mathcal{L}$. The bordism theory which uses $\wp$ - manifolds as cycles is called 'bordism with coefficients $\wp$'. A universal-coefficient theorem is proved geometrically and, as a consequence, is deduced the fact that different resolutions give rise to the same theory.

It is clear, even from this rough sketch, that the approach is an entirely geometrical one, i.e. the picture of a typical G-cycle is given, so that $h_*(-, G)$ is still a geometric theory. In the light of such geometric definition, the universal coefficient sequence may be viewed as an obstruction theory to solving the singularities of a typical G-cycle.

The definition of $h^*(-; G)$ is obtained immediately from that of $h_*(-; G)$ by considering the dual mock-bundle theory, which has a G-cycle as a typical block ([6]).

The notion of $h^*(X; F)$, where F is a sheaf over X, is given via mock bundles such that the block over a simplex $\sigma \in K$, $|K| = X$, is an $F(st(\sigma,K))$-cycle.

In the paper, for the sake of simplicity in the notations and in the phraseology, the whole description of coefficients is carried out for a particular geometric theory, i.e. oriented p.l. bordism, but everything works in general. Moreover in the cohomological case we restrict our attention to compact polyhedra; however the extension to non compact polyhedra, and therefore to all CW complexes up to homotopy type, does not present any difficulty, since mock bundles with non compact base are defined.

A geometridal description of Z_n-smooth bordism by means of manifolds with singularities has already been given by Sullivan and his Z_n-manifolds fit perfectly into the general setting considered in this paper.

I am very much indebted to my supervisor, Dr. C.P. Rourke, for continuous guidance, help and encouragement during my three years at Warwick as his student. He introduced me to the problem dealt with in this thesis and suggested the particular point of view to interpret, in the case of a short presentation, the relations as expressing the type of singularities needed in the manifolds. I also wish to express my thanks to the Italian Research Council (Consiglio Nazionale delle Ricerche) for financial support during my three years as a research student.

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C O N T E N T S

CHAPTER I: BORDISM WITH COEFFICIENTS IN AN ABELIAN GROUP

Unless otherwise stated we shall be working in the p.l. category (polyhedra and p.l. maps). For all the relevant definitions we refer to Stone [7], Ch. I.

In Sec. 1 we define bordism with coefficients in a four-term resolution $\rho \longrightarrow G$ (with extra structure) by means of singularities depending thoroughly on the chosen resolution. In Sec. 2 we prove a universal-coefficient theorem for bordism with coefficients in a resolution. The proof consists of interpreting the singularities as elements of the torsion product of the relevant groups. In sec. 3 we give a definition of bordism with coefficients in a group G by fixing a particular presentation for every G. The main problem is to prove functoriality on the morphisms, because a map of presentations does not induce a map between the corresponding theories in any obvious way. Therefore we need to factorise any map through a longer re= solution and then use the universal-coefficient theorem. This is the part where the presence of singularities in codimension greater than one plays its essential role.

1. Links and manifolds associated to a free resolution

Let G be an abelian group. A structured resolution $\wp$ of G consists of

(a) a free resolution of G of length $\leqslant 4$

$$: \; 0 \longrightarrow F_3 \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\varepsilon} G \longrightarrow 0$$

(b) a basis, B^p, for each F_p $(p = 0,1,2,3)$

(c) for each $b^p \in B^p$ $(p = 1,2,3)$ we are given an unordered word, $w(b^p)$, representing the element $_p(b^p)$. Precisely $w(b^p)$ is a finite set (of pairs)

$\{\delta_i b_i^{p-1} \mid$ i varies over a finite set $I(b^p)$, $\delta_i = +$ or $-$; $b_i^{p-1} \in B^{p-1}\}$ such that $\sum_i \delta_i b_i^{p-1} = \phi_p(b^p)$, where the sum denotes the element of F_{p-1} obtained from $w(b^p)$ by adding up all the coefficients belonging to the same element in B^{p-1} .

(d) For each $b^p \in B^p$ $(p = 2,3)$ we are given a 'cancellation rule' defined as follows. Let $w(b^p) = \{\delta_i b_i^{p-1} : i \in I(b^p)\}$. Then order two of $\wp$ implies that the formal sum $\sum_i \delta_i w(b_i^{p-1})$ is an unordered word representing the zero element of F_{p-2} in terms of the elements of B^{p-2}. The effect of δ_i is an inversion of sign iff $\delta_i =$ '$-$' .

The given cancellation rule consists of a procedure for pairing off the letters of $\sum_i \delta_i w(b_i^{p-1})$ in F_{p-2} . Precisely $c(b^p)$ is a partition of the letters of $\sum_i \delta_i w(b_i^{p-1})$ into pairs of the form $(\delta_j b^{p-2}, \delta_k b^{p-2})$ with $\delta_j \neq \delta_k$.

For the sake of simplicity we may write the set $I(b^p)$ in the form $\{1,\ldots,\ell : \ell = \ell(b^p)\}$.

If $0 \leqslant k \leqslant 3$ then $\wp_k$ is the structured resolution of $\operatorname{Im}\phi_k$

$$\wp_k : \; 0 \longrightarrow F_3 \xrightarrow{\phi_3} \ldots \longrightarrow F_k \xrightarrow{\phi_k} \operatorname{Im}\phi_k \longrightarrow 0$$

where the structure is that induced from ρ . Clearly $\rho_0 = \rho$.

For each quadruple (G, ρ, p, n), where G is an abelian group, ρ is a structured resolution of G; p, n are integers such that $-1 \leqslant p \leqslant 3$; $n \geqslant 0$, we shall construct

(a) a class $\mathcal{L}^p$ of p.l. isomorphism-types of polyhedra, called the class of (p, ρ)-links

(b) a class of p.l. isomorphism-types of polyhedra, called the class of (ρ, n)-manifolds

(c) a generalized homology theory $\{\Omega_*(-;\rho), \partial\}$

For pictures we refer to the examples given later. To construct $\mathcal{L}^p$ we proceed by induction on p. $\mathcal{L}^{-1} = \{\emptyset\}$. If p = 0, let $b^1 \in B^1$, $w(b^1) = \sum_{i=1}^{\ell} \delta_i b_i^o$. Give the set $\{b_1^o, \dots, b_\ell^o\}$ the discrete topology and each point b_i $(i=1,\dots,\ell)$ the orientation δ_i . The resulting polyhedron will be called the <u>0-link associated to b^1 in ρ</u> (or <u>generated</u> by b^1 in ρ) and will be written $L(b^1, \rho)$. We define the class $\mathcal{L}^o$ to consist of all polyhedra $L(b^1, \rho)$ with $b^1 \in B^1$, i.e. $\mathcal{L}^o \equiv \{ L(b^1, \rho) : b^1 \in B^1 \}$. Now consider the join of $L(b^1, \rho)$ and the point b^1, written $b^1L(b^1, \rho)$, and give $b^1L(b^1, \rho)/b^1$ (= the open cylinder over $L(b^1, \rho)$) the orientation $- \longrightarrow +$ (arrow departing from '-' sign). The join $b^1L(b^1, \rho)$, with the above orientation out of b^1 and with the orientation '+' on b^1, will be referred to as the (oriented) <u>cone</u> over $L(b^1, \rho)$ with vertex b^1 . In general it happens that different elements of B^1 may give rise to the same 0-link. Therefore, over the same link, there may be different cones, corresponding to different vertices generating the link.

If p = 1 , let $b^2 \in B^2$; $w(b^2) = \sum_{i=1}^{\ell(b^2)} \delta_i b_i^1$. Construct the

0-link $L(b^2, \wp_1)$ as in the previous case. Each b_i^1 generates a 0-link $L(b_i^1, \wp)$ $(i=1,\dots,\ell)$; consider the space $\bigcup_{i=1}^{\ell} \delta_i \left[b_i^1 L(b_i^1, \wp) \right] = L$ where $b_i^1 L(b_i^1, \wp)$ is the cone as defined above and δ_i changes all the orientations present in this cone iff $\delta_i =$ '-' . Let $\overline{w} = \sum \delta_i w(b_i^1) = \{ b_1^o, \dots, b_t^o$ with the appropriate signs $\}$. Then $\overline{L} = \bigcup_{i=1}^{\ell} \delta_i L(b_i^1, \wp)$ is obtained by giving $\overline{w}$ the discrete topology and orientations according to the signs. Therefore the cancellation rule $c(b^2)$ gives a canonical way of joining the points of $\overline{L}$ in pairs by plugging in oriented 1-disks. Precisely suppose $\delta_j b_j^o$ is paired with $\delta_k b_k^o$. Then $\delta_j \neq \delta_k$ and we insert a 1-disk $[b_j^o, b_k^o]$ with orientation given according to the rule 'arrow departing from '+' sign'. Moreover we label the 1-disk by the unique element $b_j^o = b_k^o \in B^o$. The object which is obtained from L through the above identifications in $\overline{L}$ is called the <u>1-link associated</u> to b^2 in $\wp$ (or generated by b^2 in $\wp$) and it is written $L(b^2, \wp)$. The class of 1-links, $\mathcal{L}^1$, is defined by $\mathcal{L}^1 \equiv \{ L(b^2, \wp) : b^2 \in B^2 \}$. (See picture I.3)

It is clear that it is the given cancellation rule in $\wp$ that makes the construction well defined. Different rules may give rise to completely different links (picture I.4).

We think of $L(b^2, \wp)$ as a one-dimensional stratified set in which the pure j-stratum (j=0,1) consists of a disjoint union of j-disks, each disk is oriented and labelled by one element of B^{1-j}; the 0-stratum is the 0-link generated by b^2 in $\wp_1$.

It is clear how to define the <u>cone $b^2 L(b^2, \wp)$</u> : topologically $b^2 L(b^2, \wp)$ is the usual cone over $L(b^2, \wp)$ with vertex b^2, the subcone over the 0-stratum of $L(b^2, \wp)$ is given an orientation outside b^2 as in the previous step; the subcone over the 1-stratum has the orientation given by the cartesian product:

(1-stratum) $\times$ [-; +) where [-; +) is the half open 1-disk oriented 'from - to +' . Finally the vertex b^2 has the orientation + . Each stratum of $b^2L(b^2,\varrho)$ is labelled in the obvious way. Now, as before, it may well happen that different elements in B^2 generate the same 1-link and therefore over the same link there may be different cones.

Now suppose W is an oriented manifold, then we can form the topological product $W \times b^2L(b^2,\varrho)$. From now on we think of the above product as having the following additional structure: three intrinsic strata, namely $W \times b^2$, $W \times L_o$, $W \times L_1$, L_j being the pure j-dimensional stratum of $L(b^2,\varrho)$ $(j= 0,1)$; a labelling on each stratum obtained from the labelling of the second factor; the product orientation on each stratum.

We are now left with the case p = 2 . Let $b^3 \in B^3$. Consider the 1-link associated to b^3 in ϱ_1 and construct a trivial normal bundle system with base $L(b^3,\varrho_1)$ as follows. If $\delta_i b_i^2$ is a vertex of $L(b^3,\varrho_1)$ then put $\delta_i L(b_i^2,\varrho)$ as the fibre at that vertex. The part of $L(b^3,\varrho_1)$ which remains unclothed consists of a disjoint union of closed 1-disks. Let D, labelled by $b^1 \in B^1$, be one of such disks. The restriction of the normal bundle to ∂D is $\partial D \times L(b^1,\varrho)$; therefore we can extend the bundle by plugging in $D \times L(b^1,\varrho)$. As a result of clothing the 1-stratum of $L(b^3,\varrho_1)$ we are left with a polyhedron, whose boundary consists of 1-spheres, labelled by elements of B^o. Then plug in an oriented labelled 2-disk for each sphere and get the required link $L(b^3,\varrho)$.

The cone $b^3L(b^3,\varrho)$ and the product $W \times b^3L(b^3,\varrho)$, W oriented manifold, are defined as in the previous cases.

Now we define $(\mathcal{L}^p,n)$-manifolds inductively on p as follows.

An <u>$(\mathcal{L}^{-1},n)$-manifold</u> is an oriented n-manifold in which each connected component is labelled by an element of B^o .

(a) $M - SM$ is a (possibly open) $(\mathcal{L}^{p-1},n)$-manifold

(b) SM is a (possibly empty) oriented manifold in which each component is labelled by an element of B^{p+1} .

(c) (Trivialized stratification) If V is a component of SM labelled by $b^{p+1} \in B^{p+1}$ an isomorphism

$$h : (N,V) \longrightarrow (V \times {}_{b^{p+1}}L(b^{p+1},\rho),\ V \times b^{p+1})$$

is given, where N is a regular neighbourhood of V in M .

An $(\mathcal{L}^2,n)$-manifold will be called a $\underline{(\rho\ ,n)\text{-manifold}}$, or simply ρ-manifold if the dimension is understood.

Remarks

1. A p-link is an $(\mathcal{L}^{p-1},p)$-manifold $(p = 0,1,2,3)$.

2. In the definition of the links and of the $(\mathcal{L}^p,n)$-manifolds we have used only order two of the resolution ρ .

We list some examples.

3. $$\rho : 0 \longrightarrow F(B) \xrightarrow{\varepsilon\, =\, id} F(B) \longrightarrow 0$$

i.e. $G = F_o$ = free abelian group over B .

A $(\rho\ ,n)$-manifold is an oriented n-manifold M, each component of which is labelled by an element of B . There are no (p,ρ)-links other than $\emptyset$.

4. $$\rho : 0 \longrightarrow F(b^1) \xrightarrow{\phi_1} F(b_1^o,b_2^o) \xrightarrow{\varepsilon} Z \longrightarrow 0$$

$\varepsilon(b^o_1) = 1 \in Z$; $\varepsilon(b^o_2) = 2$; $w(b^1) = b^o_1 + b^o_1 - b^o_2$.

A (ρ,n)-manifold is represented in picture I.1 .

5. $\rho : 0 \longrightarrow F(b^1) \xrightarrow{\phi} F(b^o) \xrightarrow{\varepsilon} Z_n \longrightarrow 0$

$w(b^1) = \underbrace{b^o + \ldots + b^o}_{\text{n-times}}$; $\varepsilon(b^o) = 1 \in Z_n$.

$L(b^1, \rho)$ is a set of n-points with orientation '+' . Therefore (ρ ,m)-manifolds are essentially Sullivan's Z_n-manifolds of dimension m .

6. $\rho : 0 \longrightarrow \mathrm{Ker}\ \phi_2 \xrightarrow{\phi_3} F\ \mathrm{Ker}\ \phi_1 \xrightarrow{\phi_2} F\ \mathrm{Ker}\ \varepsilon \xrightarrow{\phi_1} FZ_3 \longrightarrow Z_3 \longrightarrow 0$

Let $b^3 \in \mathrm{Ker}\ \phi_2$ be such that

$w(b^3) = - b^2_1 + b^2_2 + b^2_3$

$w(b^2_1) = - b^1_{11} + b^1_{12} \in F\ \mathrm{Ker}\,\varepsilon$

$w(b^2_2) = b^1_{21} - b^1_{22} + b^1_{23} \in F\ \mathrm{Ker}\ \varepsilon$

$w(b^2_3) = - b^1_{31} + b^1_{32} + b^1_{33} \in F(\mathrm{Ker}\,\varepsilon)$

$w(b^1_{11}) = b^o_1 + b^o_2$

$w(b^1_{12}) = - b^o_1 - b^o_2$

$w(b^1_{21}) = b^o_1 + b^o_2$

$w(b^1_{22}) = b^o_o + b^o_{11} + b^o_{12} + b^o_{21} + b^o_{22}$

$w(b^1_{23}) = b^o_o + b^o_1 + b^o_2$

$w(b^1_{31}) = b^o_o + b^o_1 + b^o_2$

$w(b^1_{32}) = b^o_o + b^o_{11} + b^o_{12} + b^o_{21} + b^o_{22}$

$w(b^1_{33}) = - b^o_1 - b^o_2$

$b^o_{11} = b^o_{12} = b^o_1 \in FZ_3$; $b^o_{21} = b^o_{22} = b^o_2 \in FZ_3$; $\varepsilon(b^o_o) = 0 \in Z_3$;

$\varepsilon(b^o_1) = 1 \in Z_3$; $\varepsilon(b^o_2) = 2 \in Z_3$

Picture I.2 shows the construction of $L(b^3, \rho)$, the cancellation rules being suggested by the picture itself.

7. $\rho : 0 \longrightarrow \mathrm{Ker}\ \phi_2 \longrightarrow F\ \mathrm{Ker}\ \phi_1 \xrightarrow{\phi_2} F\ \mathrm{Ker}\ \varepsilon \xrightarrow{\phi_1} FZ_5 \longrightarrow Z_5 \longrightarrow 0$

Let b^2 be a basis element of $F\ \mathrm{Ker}\ \phi_1$. Suppose

$$w(b^2) = b_1^1 + b_2^1 + b_3^1 + b_4^1$$

$$w(b_1^1) = b_{11}^o + b_{12}^o + b_{13}^o$$

$$w(b_2^1) = - b_{21}^o + b_{22}^o + b_{23}^o$$

$$w(b_3^1) = - b_{31}^o - b_{32}^o + b_{33}^o$$

$$w(b_4^1) = - b_{41}^o - b_{42}^o - b_{43}^o$$

$$\varepsilon(b_{11}) = \varepsilon(b_{43}^o) = 2\ ; \quad \varepsilon(b_{12}^o) = \varepsilon(b_{42}^o) = 3\ ;$$

$$\varepsilon(b_{13}^o) = \varepsilon(b_{21}^o) = \varepsilon(b_{33}^o) = \varepsilon(b_{41}^o) = 0\ ;$$

$$\varepsilon(b_{22}^o) = \varepsilon(b_{32}^o) = 1\ ; \quad \varepsilon(b_{23}^o) = \varepsilon(b_{31}^o) = 4\ .$$

Picture I.3 shows two possible links associated to b^2 in ρ, corresponding to different cancellation rules.

8. $: 0 \longrightarrow F(b_1^1, b_2^1, \ldots) \xrightarrow{\phi_1} F(b_o^o, b_1^o, \ldots) \longrightarrow Z[1/2] \longrightarrow 0$

$Z[1/2]$ is the additive group of those rational numbers whose denominators are powers of 2 .

$$\varepsilon(b_i^o) = \frac{1}{2^i} \quad (i = 0,1,\ldots)\ , \quad w(b_j^1) = b_j^o + b_j^o - b_{j-1}^o\ .$$

A (ρ, n)-manifold M has two intrinsic strata in dimensions n and n-1 . Each component V of the singular stratum is labelled by an element $b_j^1 = \frac{1}{2^j} + \frac{1}{2^j} - \frac{1}{2^{j-1}}$ and there are three sheets merging into V ; two of them are labelled by $\frac{1}{2^j}$, the third is labelled by $\frac{1}{2^{j-1}}$. The sheets are oriented and the normal bundle of V in M is trivialized.

The systems of labellings, orientations and trivializations on a ($\wp$,n)-manifold M will be referred to as the <u>(extra)-structure</u> of M. If M, M' are two ($\wp$,n)-manifolds, an <u>isomorphism</u> $f: M \cong M'$ is a homeomorphism, which preserves the whole structure. There is a notion of <u>boundary</u>, ∂M, of a (compact) ($\wp$,n)-manifold M :

(a) topologically ∂M is the subset of M which is maximal with respect to the properties of being closed and locally collared in W.

(b) ∂M has a structure of ($\wp$,n-1)-manifold obtained from that of M by restriction.

A ($\wp$,n)-manifold M is said to be <u>closed</u> if $\partial M = \emptyset$. Let M be a ($\wp$,n)-manifold and M' a ($\wp$,n')-manifold. An <u>embedding</u> $f: M' \longrightarrow M$ is a locally flat stratified embedding between the underlying polyhedra, which is compatible with the labelling and the trivialisations. If n' = n, the f may be orientation preserving or orientation-reversing. In the following, unless otherwise stated, a codimension zero embedding will always be assumed to be orientation preserving. A <u>submanifold</u> of a ($\wp$,n)-manifold M is a subset $M_o \subset M$ together with an embedding $f: M'_o \hookrightarrow M$ (of $\wp$-manifolds) such that $f(M') = M$. If M is a ($\wp$,n)-manifold, -M denotes the ($\wp$,n)-manifold obtained from M by reversing all the orientations; $(\wp, n)$-manifolds satisfy the following axioms.

9. If M is a $(\wp, n)$-manifold, $M \times I$ has a natural structure of $(\wp, n+1)$-manifold, obtained by crossing the structure of M with I, it is clear that $\partial (M \times I) \cong M \cup -M \cup \partial M \times I$.

10. If M, M' are $(\wp, n)$-manifolds and M_o, M'_o are $(\wp, n-1)$-submanifolds of ∂M, $\partial M'$ respectively such that $M_o \overset{g}{\cong} -M'_o$, then $M \cup_g M'$ is a $(\wp, n)$-manifold with boundary isomorphic to $Cl(\partial M/M_o \cup_g \partial M'/M'_o)$.

11. Let M be a $(\wp, n)$-manifold and $M_o \subset M$. Suppose $Fr(M_o)$ is collared in M_o and bicollared in M. Then M_o is a $(\wp, n)$-manifold with $\partial M_o = \partial M \cap M_o \cup Fr(M_o)$.

The proofs of 9, 10, 11 are omitted because they use standard geometrical arguments, which are easily checked to be compatible with the structure that we have on our objects.

Now fix a pair X, A of topological spaces. A <u>singular</u> $(\wp, n)$-manifold in X, A is a pair (M, f) consisting of a $(\wp, n)$-manifold M and a map $f: (M, \partial M) \longrightarrow (X, A)$. A singular $(\wp, n)$-manifold (M, f) is said to <u>bord</u> if and only if there exists a $(\wp, n+1)$-manifold W and a map $F : W \longrightarrow X$ for which

(a) M is a submanifold of ∂W

(b) $F|M = f$ and $F(\partial W/M) \subset A$.

Define $-(M, f) = (-M, f)$. Two singular (ρ, n)-manifolds (M_1, f_1), (M_2, f_2) are <u>bordant</u> iff the disjoint union $(M_1 \cup -M_2, f_1 \cup f_2)$ bords in X, A. From the properties 9, 10 of (ρ, n)-manifolds it follows immediately that bordism is an equivalence relation in the set of singular (ρ, n)-manifolds in (X, A). Denote the bordism class of (M, f) by $[M, f]_\rho$ and the set of all such bordism classes by $\Omega_n(X; A; \rho)$. An abelian group structure is given in $\Omega_n(X; A; \rho)$ by disjoint union, i.e. $[M_1, f_1]_\rho + [M_2, f_2]_\rho = [M_1 \cup M_2, f_1 \cup f_2]_\rho$. The class of all (M, f) that bord forms the zero element and $-[M, f]_\rho = [-M, f]_\rho$. We refer to $\Omega_n(X; A; \rho)$ as the <u>n-dimensional (oriented) bordism group of X, A with coefficients in</u> ρ. When no confusion arises, we shall drop the subscript ρ in the notation $[\quad]_\rho$. Given a map $h : (X, A) \longrightarrow (X', A')$, there is associated a homomorphism $h_*: \Omega_n(X; A; \rho) \longrightarrow \Omega_n(X'; A'; \rho)$ by $[M, f] = [M, hf]$, which makes $\Omega_n(-; -; \rho)$ into a functor. There is also a 'boundary' homomorphism $\partial_n : \Omega_n(X, A; \rho) \to \Omega_{n-1}(A, \emptyset; \rho)$ by $\partial [M, f] = [\partial M, f|\partial M]$.

12. <u>Proposition</u> The graded functor $\{\Omega_n(X, A; \rho), \partial_n\}$ is a generalised homology theory on the category of topological spaces.

The proposition is a direct consequence of the axioms 9, 10, 11 and the proof is therefore omitted. □

The following example will be referred to later.

13. Take G = F = free abelian group over B and ρ to be the resolution $0 \longrightarrow F(B) \xrightarrow{id} F(B) \longrightarrow 0$. We want to prove that in this case the group $\Omega_n(X; A; \rho)$ is isomorphic to $\Omega_n(X; A) \otimes F$ for every n and

(X, A). Here $\Omega_n(X, A)$ denotes the usual oriented n-bordism group of the pair X, A. For simplicity we assume $(X,A) = (\text{point}, \emptyset)$ and write $\Omega_n(pt; \emptyset, \rho) = \Omega_n(\rho)$. An element of $\Omega_n(\rho)$ is, by definition, a bordism class represented by an oriented manifold M, each component of which is labelled by an element of B (see Example 3). Denote such bordism class by $[M]_{\otimes F}$. We can write $M = \cup_i(V_i, b_i)$, where V_i is the union of all the components of M labelled by $b_i \in B$. In the above notations we give a map $h: \Omega_n(\rho) \longrightarrow \Omega_n \otimes F$ by $h : [M]_{\otimes F} \longmapsto \sum_i [V_i] \otimes b_i$. It is an easy exercise to prove that h is a well defined isomorphism of groups. By means of h the operation of tensoring a (bordism class of a) manifold by an element of B may be interpreted as a labelling operation. In the following we shall deliberately confuse $h([M]_{\otimes F})$ with $[M]_{\otimes F}$ and omit the subscript $\otimes F$ when no misunderstanding is possible.

2. The Universal-Coefficient Sequence

We have constructed a homology theory $\{\Omega_n(X; A; \rho). \partial_n\}$, associated to the resolution ρ of G. Now we are going to prove the following.

1. Proposition For each integer $n \geq 0$ and each pair X, A there exists a short exact sequence

$$0 \longrightarrow H_0(\rho, \Omega_n(X; A)) \xrightarrow{1} \Omega_n(X; A; \rho) \xrightarrow{s} H_1(\rho, \Omega_{n-1}(X,A)) \longrightarrow 0$$

which is natural in X, A.

Proof: First of all we anticipate that the proof consists of geometrical arguments involving only (ρ, n)-manifolds and their stratifications. In the constructions, the maps into X, A do not play an essential role and so, for the sake of simplicity, we shall assume $(X, A) = (\text{point}, \emptyset)$.

Description of the map s. It involves a resolution of singularities. Let M be a closed (φ, n)-manifold. We are going to show that there exists a (φ, n)-manifold $\tilde{M}$, bordant to M and having no singularities in codim > 1. Let SM be the last stratum of M. Then, by definition of (φ, n)-manifold, each connected component V_i of SM is a manifold, oriented and labelled by an element $b_i \in B^p$, if SM has dimension n - p. If p = 1, there is nothing to prove. Suppose p > 1. Then, by example 1.19, $[SM]_{\otimes F_p}$ can be identified with an element of $\Omega_{n-p} \otimes F_p$, i.e. $\sum_i [V_i] \otimes b^p_i$. We need the following

2. Lemma Suppose that $[SM]_{\otimes F_p} = [S]_{\otimes F_p}$. Then M is bordant to a (φ, n)-manifold Q, whose last stratum is S, by a bordism R whose last stratum is still in codimension p.

Proof: Consider the following spaces:

$M \times I$, where $I = [0, -1]$;

SR = any bordism between -SM and -S:

assume that SR consists of a set of equally labelled components, with label, say, $b^p \in B^p$;

$\nu(-SM)$ = normal bundle of -SM in $-M = M \times \{-1\}$;

$SR \times L(b^p, \varphi)$

We have: $\nu(-SM) \subset M \times \{-1\}$; $\nu(-SM) \subset SR \times L(b^p, \varphi)$. So we can form the identification space:

$$R = SR \times L(b^p, \varphi) \underset{\nu(-S(M))}{\smile} M \times I$$

which provides the required bordism.

If SR has many labelling elements, we perform the above construction simultaneously on every set of equally labelled components. The last stratum of R is SR and has codimension p. $\square$

3. Remark If we can choose $S = \emptyset$ then the above construction gives a resolution of the low dimensional singularities of M. In other words, when the low dimensional singularities of a (ρ, n)-manifold M bound (in a labelled sense), they can be resolved by means of a bordism having the same kind of singularities as M.

Proof of Proposition 1 (continued). Now let us look at the image of $[SM]_{\otimes F_p}$ through the morphism

$$\tilde{\phi}_p : \Omega_{n-p} \otimes F_p \xrightarrow{id \otimes \phi_p} \Omega_{n-p} \otimes F_{p-1}.$$

We have, for each component $V_i \otimes b^p{}_i \subset SM$, $\tilde{\phi}_p([V_i] \otimes b_i) = [V_i] \otimes w(b^p{}_i) = [V_i] \otimes \sum_j \delta_j b_j{}^{p-1} = \sum_j [\delta_j V_i] \otimes b_j^{p-1}$: this is nothing else than the bordism class of the boundary of the complement of a regular neighbourhood of V_i in the $(n - p + 1)$-stratum of M. Therefore the image of $[SM]_\otimes$ is the bordism class of the boundary of the complement of a regular neighbourhood of SM in the $(n - p + 1)$-stratum of M and, as such, is the zero element of $\Omega_{n-p} \otimes F_{p-1}$. Now, because $p > 1$, the sequence:

$$\Omega_{n-p} \otimes F_{p+1} \xrightarrow{\tilde{\phi}_{p+1}} \Omega_{n-p} \otimes F_p \xrightarrow{\tilde{\phi}_p} \Omega_{n-p} \otimes F_{p-1}$$

is exact and so there exists an element $[SW]_{\otimes F_{p+1}} \in \Omega_{n-p} \otimes F_{p+1}$ such that $\tilde{\phi}_{p+1}[SW]_\otimes = [SM]_{\otimes F_p}$. Suppose first that SW is a set of components all labelled by $b^{p+1} \in B^{p+1}$. We can always reduce to the case $SM = SW \otimes w(b^{p+1})$, because if SM is only bordant to $SW \otimes w(b^{p+1})$, $SM \sim SW \otimes w(b^{p+1})$, then, by Lemma 2, M can be replaced by another (ρ, n)-manifold $\bar{M}$ such that:

(a) $\bar{M}$ is bordant to M by means of a bordism with singularities up to codimension p only

(b) $S\bar{M} = SW \otimes w(b^{p+1})$.

Therefore assume $SM = SW \otimes w(b^{p+1})$. If $SW \sim \emptyset$ we are reduced to the case of Remark 3 and we know how to solve the singularities then. So assume $SW \neq \emptyset$ and take the following spaces (Picture I.6):

$$M \times I, \text{ where } I = [0, -1]$$
$$-(SW \times b^{p+1} L(b^{p+1}, \varrho))$$
$$\nu(-SM) = \text{normal bundle system of } -SM \text{ in } M \times \{-1\} = -M$$

Then we have $\nu(-SM) \subset M \times I$ and $\nu(-SM) \subset -(SW \times b^{p+1} L(b^{p+1}, \varrho))$. The identification space

$$W = -(SW \times b^{p+1} L(b^{p+1}, \varrho)) \underset{\nu(-S(M)}{\smile} M \times I$$

realises a bordism between $M = M \times \{0\}$ and a (ϱ, n)-manifold M' whose last stratum has dimension $n - p + 1$. The singularities SM have been resolved up to bordism. In general, if SW is of the form $SW = \Sigma_k (SW)_k \otimes b_k^{p+1}$, then one performs the above construction simultaneously on all terms $(SW)_k \otimes b_k^{p+1}$ and gets the desired manifold M'.

We remark that in order to get rid of singularities in codimension p we have used bordisms, which have singularities up to codimension p + 1.

So now we have a well defined procedure to solve the singularities of a (ϱ, n)-manifold M, stratum by stratum, starting from the last one and going up by one dimension each time, until we are left with a (ϱ, n)-manifold, $\widetilde{M}$, which is bordant to M and has singularities $S\widetilde{M}$ in codimension one at most. In general we cannot solve $S\widetilde{M}$ as above, because the sequence

$$\Omega_{n-1} \otimes F_2 \xrightarrow{\widetilde{\Phi}_2} \Omega_{n-1} \otimes F_1 \xrightarrow{\widetilde{\Phi}_1} \Omega_{n-1} \otimes F_0$$

is not necessarily exact. However $\widetilde{\Phi}_1 [S\widetilde{M}]_{\otimes F_1} = 0$ in $\Omega_{n-1} \otimes F_0$ because it is the bordism class of the boundary of the complement of a regular neighbourhood of $S\widetilde{M}$ in $\widetilde{M}$.

The next lemma is important in what follows.

4. Lemma Suppose that M is a $(\rho, n+1)$-manifold with boundary ∂M. Then, if ∂M has singularities up to codimension p and M has singularities up to codimension $p + h$, there is a (ρ, n)-bordism with boundary, $W, \partial W$, between $M, \partial M$ and $N, \partial N$ such that

(a) W has singularities up to codimension $p + h + 1$ at most

(b) ∂W has, singularities up to codimension p

(c) N has singularities up to codimension $p + 1$.

Proof. The intrinsic stratum of M in codimension $p + 1$ is a closed polyhedron, because the singularities of ∂M are in codimension p at most. Therefore the above construction for solving the singularities can be applied, essentially unaltered, to solve the codimension $p + 2$ stratum of $M, \partial M$ by a bordism with boundary, $W, \partial W$. No new singularities are created along the boundary during this process. Hence lemma follows. □

Proof of Proposition 2 (continued): Each $M \in [M]_\rho$ determines an element of $\operatorname{Ker} \phi_1$ and precisely $[S\widetilde{M}]_{\otimes F_1}$. But such assignment depends on the representative M; in fact, if $M_1 \sim M$, then $S\widetilde{M}_1$ may be bordant to $S\widetilde{M}$ by means of a bordism, V, with singularities and therefore in general $[S\widetilde{M}_1]_{\otimes F_1} \neq [S\widetilde{M}]_{\otimes F_1}$. However, by Lemma 4, the singularities of V can be assumed to have codimension one at most and we can certainly write: $[S\widetilde{M}]_{\otimes F_1} = [S\widetilde{M}_1]_{\otimes F_1} + [N]_{\otimes F_1}$ where $[N]_{\otimes F_1} \in \operatorname{Im} \widetilde{\phi}_2$; more precisely: $[N]_{\otimes F_1} = \widetilde{\phi}_2[SV]_{\otimes F_2}$. Thus there is a well defined map:

$$s : \Omega_n(\rho) \longrightarrow H_1(\rho, \Omega_{n-1})$$

$$s : [M] \longmapsto [S\widetilde{M}]_{\otimes} + \operatorname{Im} \widetilde{\phi}_2$$

It is straightforward to check that s is a morphism of groups.

s is a epimorphism. Take $[V]_{\otimes} + \operatorname{Im} \widetilde{\phi}_2 \in H_1(\rho, \Omega_{n-1})$; $\widetilde{\phi}_1[V]_{\otimes} = 0$. Suppose V constantly labelled by $b^1 \in B^1$; then $V \otimes w(b^1)$ bounds in $\Omega_{n-1} \otimes F_o$, i.e. $V \otimes w(b_1) = \partial \widetilde{V}$. Take a copy of V and label it by b^1, $V \otimes b^1$; $\partial \widetilde{V}$ consists of a number of copies of V (non constantly labelled in general); identify each copy with $V \otimes b^1$. $\widetilde{V}$, with the above identification on its boundary, becomes a (ρ, n)-manifold $\widetilde{W}$ with singularities in codimension one only. Therefore, to each manifold V, representing an element in $H_1(\rho, \Omega_{n-1})$ we are able to associate a (ρ, n)-manifold $\widetilde{W}$, representing an element in $\Omega_n(\rho)$, such that $s[\widetilde{W}]_{\rho} = [V]_{\otimes} + \operatorname{Im} \widetilde{\phi}_2$. In fact, if $\widetilde{W}' \in [\widetilde{W}]_{\rho}$ we can assume, by lemma 4, that $\widetilde{W}'$ has singularities $S\widetilde{W}'$ in codimension at most one and that there exists a bordism $N : \widetilde{W}' \sim \widetilde{W}$ with singularities SN in codimension two at most. Then $[S\widetilde{W}]_{\otimes} - [S\widetilde{W}']_{\otimes} = \widetilde{\phi}_2[SN]_{\otimes}$ and so s is an epimorphism.

Description of the map l. Define a map $\widetilde{l} : \Omega_n \otimes F_o \longrightarrow \Omega_n(\rho)$ by $\widetilde{l}[M]_{\otimes F_o} = [M]_{\rho}$; $\widetilde{l}$ is a well defined homomorphism; so we have the sequence $\Omega_n \otimes F_1 \xrightarrow{\widetilde{\phi}_1} \Omega_n \otimes F_o \xrightarrow{\widetilde{l}} \Omega_n(\rho)$. $\widetilde{l}\widetilde{\phi}_1 = 0$ because let $[W]_{\rho} = \widetilde{l}\widetilde{\phi}_1[M]_{\otimes F_1}$ and suppose M constantly labelled by $b^1 \in B^1$. Then take $M \times b^1 L(b^1, \rho)$ and observe that $\partial(M \times b^1 L(b^1, \rho))$ is bordant to W. So stick the two bordisms together and get a bordism of W to the empty set by means of a$(\rho, n+1)$-manifold with codimension one singularities.

Assume now $\widetilde{l}([M]_{\otimes}) = 0$. Pick a representative V of $\widetilde{l}([M]_{\otimes})$: there exists a bordism, N, of V to $\emptyset$ such that N has singularities SN in codimension one at most. We claim that $\widetilde{\phi}_1[SN]_{\otimes F_1} = [M]_{\otimes F_o}$. In fact remove from N a regular neighbourhood of SN in N to get the required bordism between M and $\widetilde{\phi}_1(SN)$. Thus we have proved that the sequence ~~above is exact,~~ ~~which is~~ enough to ensure the existence of a monomorphism

$l : H_o(\rho, \Omega_n) \longrightarrow \Omega_n(\rho)$ induced by $\tilde{l}$.

Now it only remains to prove exactness at $\Omega_n(\rho)$.

$sl = 0$: $s\tilde{l}[M]_\otimes = 0$, because $[M]_\otimes$ has no singularities; hence $sl = 0$.

Ker $s \subset$ lml : let $[M]_\rho \in \Omega_n(\rho)$ and assume, without loss of generality, that M has codimension one singularities SM; $s[M]_\rho = 0$ means that $[SM]_{\otimes F_1} \in$ lm $\tilde{\phi}_2$. But then SM can be solved up to bordism; therefore $[M]_\rho = [M']_\rho$ where M' is without singularities and so it determines an element of $\Omega_n \otimes F_o$ whose image through $\tilde{l}$ is $[M]_\rho$. Thus Ker s $\subset$ lm $\tilde{l}$ = lm l.

The proof of the proposition is now complete. □

5. Remark We have seen how the exactness of ρ is used in the proof of the universal-coefficient theorem.

As pointed out before, if ρ is any based ordered chain complex augmented over G, then the theory $\Omega_*(-, \rho)$ can be defined in the same way. But now the singularities in codimension greater than one are not necessarily solvable; they give rise to the E^2-term of a spectral sequence running.

$$H_p(\rho, \Omega_q(-)) \Longrightarrow \Omega_*(-; \rho)$$

Such spectral sequence collapses to the universal coefficient formula when ρ is exact.

3. Functoriality

The classes of links constructed in Section 1 summarize the whole structure of the resolution ρ geometrically. Therefore, from now on, we refer to ρ as to a linked resolution.

If ρ, ρ' are linked resolutions of G, G' respectively, a chain map $f: \rho \longrightarrow \rho'$ is said to be a map of linked resolutions (or simply linked map) if

(a) f is based, i.e. $f(B) \subset B'^p$

Let $b^p \in B^p$. We can relabel each stratum of a link $L(b^p, \rho)$ according to f. If $f(L(b^p, \rho))$ denotes the resulting object, we require

(b) $f(L(b^p, \rho)) = L(fb^p, \rho')$ (i.e. f is link-preserving).

So there is a category, $\mathcal{C}$, whose objects are linked resolutions $\rho \xrightarrow{\varepsilon} G$ and whose morphisms are linked maps. If $\mathcal{Ab}_*$ is the category of abelian groups, we have the following

1. Proposition $\Omega_*(X,A;\rho)$ is a functor $\mathcal{C} \longrightarrow \mathcal{Ab}_*$.
(For the sake of simplicity we disregard the topological component of $\Omega_*(-;-)$).

Proof Let $\tau : \rho \longrightarrow \rho'$ be a morphism of $\mathcal{C}$. If $[M]_\rho \in \Omega_n(-;\rho)$, we associate a (ρ',n)-manifold, $\tau(M)$, to M by relabelling all the strata of M according to the based map τ. The correspondence $[M]_\rho \longrightarrow [\tau(M)]_{\rho'}$ gives a well defined natural transformation of theories $\tau_*: \Omega_*(-;\rho) \longrightarrow \Omega_*(-;\rho')$ and the functorial properties are clear. $\square$

2. <u>Corollary</u> If the linked map $\tau : \rho \longrightarrow \rho'$ is a homotopy equivalence, then τ_* is an isomorphism.

<u>Proof</u> It is an easy consequence of the universal-coefficient theorem. There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_o(\rho, \Omega_n(X,A)) \arrow[r] \arrow[d, "{H_o(\tau, \Omega_n(X,A))}"'] & \Omega_n(X,A;\rho) \arrow[r] \arrow[d, "\tau_*"] & H_1(\rho, \Omega_{n-1}(X,A)) \arrow[r] \arrow[d, "{H_1(\tau, \Omega_{n-1}(X,A))}"'] & 0 \\
0 \arrow[r] & H_o(\rho', \Omega_n(X,A)) \arrow[r] & \Omega_n(X,A;\rho) \arrow[r] & H_1(\rho, \Omega_{n-1}(X,A)) \arrow[r] & 0
\end{tikzcd}$$

in which the side-morphisms are isomorphisms, because τ is a homotopy equivalence. Therefore τ_* is also an isomorphism. $\square$

If $G \in \mathcal{A}b$, a <u>truncated linked (t.l.) resolution of G</u> is an exact sequence

$$\theta \ : F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_o \xrightarrow{\varepsilon} G \longrightarrow 0$$

satisfying conditions (a), (b), (c), (d) in the definition of structured resolution given in Section 1 . A linked map between t.l. resolutions is defined as in the non-truncated case and there is a category, $\tilde{\mathcal{C}}$, whose objects are t.l. resolutions and whose morphisms are linked maps.

In the following, for each $\rho \in \tilde{\mathcal{C}}$ and each topological pair X, A, we construct a graded abelian group $\{\tilde{\Omega}_*(X,A;\rho)\}$ which is a functor on $\mathcal{P} \times \mathcal{A}b$ ($\mathcal{P}$ = category of topological pairs). Fix a $\rho' \in \mathcal{C}$ obtained from ρ by choosing a based kernel of ϕ_2 .

A <u>singular (ρ,n)-cycle</u> in X, A is a pair (M,f) consisting of a (ρ',n)-manifold M and a map $f\colon (M,\partial M) \longrightarrow X,A$ such that M has at most two intrinsic strata. A singular (ρ,n)-cycle M,f is said to <u>bord</u> if there exists a $(\rho',n+1)$-manifold W and a map $F\colon W \longrightarrow X$ for which

(a) M is a submanifold of ∂W

(b) $F|M = f$ and $F(\partial W/M) \subset A$

(c) M has at most three intrinsic strata (labelled by elements of B^0, B^1, B^2)

W is called a <u>bordism</u> . Define $-(M,f) = (-M,f)$. Two singular (ρ,n)-cycles (M_1,f_1), (M_2,f_2) are <u>bordant</u> if the disjoint union $(M_1 \cup -M_2\,,\ f_1 \cup f_2)$ bords in X,A . Bordism is an equivalence relation in the set of singular (ρ,n)-cycles of X,A . Denote the bordism class of (M,f) by [M,f] and the set of all such bordism classes by $\tilde{\Omega}_n(X,A;\rho)$. An abelian-group structure is given in $\tilde{\Omega}_n(X,A;\rho)$ by disjoint union. Obviously the above definition of $\tilde{\Omega}_n(X,A;\rho)$ is independent of the chosen $\rho' \in \mathscr{C}$.

Now we fix our attention on a particular t.l. resolution, called the <u>canonical resolution</u> of G and written γ . It is defined as follows:

$$\begin{tikzcd} \gamma : \Gamma_2 \arrow[rr, "\phi_2"] \arrow[dr, "\psi_2"'] & & \Gamma_1 \arrow[rr, "\phi_1"] \arrow[dr, "\psi_1"] & & \Gamma_0 \arrow[r, "\varepsilon"] & G \arrow[r] & 0 \\ & F\,\mathrm{Ker}\,\phi_1 \arrow[ur, "\partial_2"'] & & F\,\mathrm{Ker}\,\varepsilon \arrow[ur, "\partial_1"'] & & & \end{tikzcd}$$

where

Γ_0 , F Ker ε , F Ker ϕ_1 are the free abelian groups on G , Ker ε , Ker ϕ_1 respectively

∂_1 is the obvious map; $\Gamma_1 = F\,B^1$

$B^1 = \{ (f, w(f)) \mid f \in \operatorname{Ker} \varepsilon \subset F \operatorname{Ker} \varepsilon$ and $w(f)$ is a word expressing $\partial_1 f$ in terms of the elements of $G \subset \Gamma_0 \}$;

$\psi_1(f, w(f)) = f$ and $\phi_1 = \partial_1 \psi_1$

$\Gamma_2 = F\,B^2$; $B^2 = \{ h;\ w(h),\ c(h) \mid h \in \operatorname{Ker} \phi_1$, $w(h)$ is a word expressing $\partial_2(h)$, $c(h)$ is a cancellation rule associated to $h \}$.

ψ_2 and ϕ_2 are defined similarly

γ has canonical bases G, B^1, B^2 and a canonical structure in which $(h, w(h), c(h))$ has $(w(h), c(h))$ as its structure.

3. <u>Lemma</u> If $\varphi : G \longrightarrow G'$ is a homomorphism of abelian groups, $\rho \xrightarrow{\varepsilon} G$ is a t.l. resolution of G, γ' is the canonical resolution of G' ; then φ extends to a linked map $\widetilde{\varphi} : \rho \longrightarrow \gamma'$ in a canonical way.

<u>Proof</u> Let $\rho = \{ F_p, \phi_p, \varepsilon \}$, $\gamma' = \{ \Gamma'_p, \phi'_p, \varepsilon' \}$. We proceed by induction on p . Write $(\widetilde{\varphi}) = (\widetilde{\varphi}_0, \widetilde{\varphi}_1, \widetilde{\varphi}_2)$. For $p = 0$, put $\widetilde{\varphi}_0(b^0) = \varphi\, \varepsilon\, (b^0)$ for each $b^0 \in B^0$. Inductively, let $b^p \in B^p$. Then $\widetilde{\varphi}_{p-1} \phi_p(b^p) \in \operatorname{Ker} \phi'_{p-1}$ and therefore it determines a basis element, b'^p, in $F \operatorname{Ker} \phi'_{p-1}$; b'^p has a canonical word $w(b'^p)$ and cancellation rule $c(b'^p)$ induced from those of b^p through the map $(\widetilde{\varphi}_{p-1}, \widetilde{\varphi}_{p-2})$. Therefore the assignment $b^p \longrightarrow (b'^p, w(b'^p), c(b'^p))$ defines $\widetilde{\varphi}_p$ with the required properties. $\square$

4. <u>Lemma</u> $\widetilde{\Omega}_*(X, A; \rho)$ gives a functor $\mathcal{P} \times \widetilde{\mathcal{C}} \longrightarrow \mathcal{Ab}_*$.

<u>Proof</u> Functoriality on $\mathcal{P}$ is obvious.

If $\tau : \rho \longrightarrow \rho'$ is a morphism of $\widetilde{\mathcal{C}}$ and $[M]_\rho \in \widetilde{\Omega}_n(-;\rho)$, let $\tau(M)$ be the (ρ', n)-manifold associated to M by relabelling each stratum according to τ. The correspondence $[M]_\rho \longrightarrow [\tau(M)]_\rho$ gives the required natural transformation

$$\tau_* : \widetilde{\Omega}_*(-;\rho) \longrightarrow \widetilde{\Omega}_*(-;\rho') . \quad \square$$

5. Corollary $\widetilde{\Omega}_*(X,A;\gamma)$ gives a functor $\mathcal{P} \times \mathcal{Ab} \longrightarrow \mathcal{Ab}_*$.

Proof Functoriality on $\mathcal{P}$ is obvious.

To each $G \in \mathcal{Ab}$ assign $\widetilde{\Omega}_*(X,A;\gamma)$ where γ is the canonical resolution of G; to each morphism $\varphi : G \longrightarrow G'$, $\varphi \in \mathcal{Ab}$, assign the homomorphism $\widetilde{\varphi}_* : \widetilde{\Omega}_*(X,A;\gamma) \longrightarrow \widetilde{\Omega}_*(X,A;\gamma')$, where $\widetilde{\varphi}$ is the canonical extension of φ described in Lemma 3 and $\widetilde{\varphi}_*$ is the induced homomorphism described in Lemma 4 . $\square$

In view of the previous corollary we shall write $\widetilde{\Omega}_*(X,A;G)$ instead of $\widetilde{\Omega}(X,A;\gamma)$ and φ_* instead of $\widetilde{\varphi}_*$. A (γ,n)-cycle [bordism] will also be called a <u>(G,n)-manifold</u> [<u>(G,n)-bordism</u>] .

A linked resolution of an abelian group G

$$\rho : 0 \longrightarrow F_3 \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\varepsilon} G \longrightarrow 0$$

is said to be <u>p-canonical</u> $(1 \leqslant p \leqslant 3)$ if

(a) $F_{p-1} \longrightarrow \dots \xrightarrow{\varepsilon} G \longrightarrow 0$ is the canonical resolution of G of length p-1 , i.e. $F_i = \Gamma_i$, $0 \leqslant i \leqslant p-1$, and the morphisms ϕ_i are the same as in the definition of γ.

(b) $F_{p+1} = F_{p+2} = 0$.

Let $G \in \mathcal{Ab}$ and $\rho_G \in \mathcal{C}$ a short linked resolution of G . (i.e.

$F_2 = F_3 = 0$) . Then we have the homology theory $\{\Omega_*(X,A;\wp_G),\partial\}$ defined in Section 1 . We also have the graded functor $\widetilde{\Omega}_*(X,A;G)\colon \mathcal{P} \longrightarrow \mathcal{A}b_*$ constructed at the beginning of this section. It follows from Lemma 3 that there is a canonical extension of the id: $G \longrightarrow G$ to a linked map $\widetilde{\mathrm{id}}\colon \wp_G \longrightarrow \gamma$. The latter induces a natural transformation of functors $t_G\colon \Omega_*(X,A;\wp_G) \longrightarrow \widetilde{\Omega}_*(X,A$ obtained by relabelling the $\wp_G$-manifolds according to $\widetilde{\mathrm{id}}$. The next theorem is the main step towards functoriality.

6. <u>Theorem</u> $t_G : \Omega_*(X,A;\wp_G) \longrightarrow \widetilde{\Omega}_*(X,A;G)$ is a natural equivalence of functors.

<u>Proof</u> Let $\wp_i$ be an i-canonical resolution for i = 2,3 . Then there is a commutative diagram

$$\begin{tikzcd}
\widetilde{\Omega}_*(X,A;G) \arrow[dr, "\alpha"'] & \Omega_*(X,A;\wp_G) \arrow[l, "t_G"'] \arrow[r, "t_{1,2}"] \arrow[d, "t_{1,3}"] & \Omega_*(X,A;\wp_2) \arrow[dl, "t_{2,3}"] \\
 & \Omega_*(X,A;\wp_3) &
\end{tikzcd}$$

where $t_{i,j}$ and α are the natural transformations obtained in the usual way by relabelling the cycles according to the canonical liftings of id: $G \longrightarrow G$. By Corollary 2 $t_{i,j}$ is an isomorphism $(1 \leqslant i < j \leqslant 3)$. Therefore, in order to prove the theorem, we only need to show that α is an isomorphism.

α is an epimorphism: it follows from commutativity and the fact that $t_{1,3}$ is epi.

α is a monomorphism: let M^n be a (singular) G-manifold such that $\alpha(M^n) \sim \emptyset$ in $\Omega_n(X,A;\wp_3)$. Then M^n determines an element $[M]_{\wp_2} \in \Omega_n(X,A;\wp_2)$ such that $t_{2,3}[M]_{\wp_2} = 0$. Since $t_{2,3}$ is a monomorphism, we deduce that there is a ρ_2 bordism $W: M^n \overset{\rho_2}{\sim} \emptyset$. Finally we observe that $\rho_2 \hookrightarrow \gamma$ is a linked embedding of resolutions and therefore W provides the required bordism of M^n to zero in $\widetilde{\Omega}_n(X,A;G)$.

The proof of the theorem is now complete. $\square$

We are now able to state the theorem asserting the possibility of making bordism with coefficients in a short linked resolution $\rho \xrightarrow{\varepsilon} G$ depend functorially on G .

7. Theorem

(a) There exists a (graded) functor $\Omega_*(X,A;G): \mathcal{P} \times \mathcal{Ab} \to \mathcal{Ab}_*$ which associates to each pair $(X,A;G)$ the graded abelian group $\Omega_*(X,A;\rho_G)$ where ρ_G is a fixed linked presentation of G; to every morphism $(f,\varphi) : (X,A,G) \to (X',A',G')$ the graded homomorphism $(f_*, t_{G'}^{-1}\varphi_* t_G) : \Omega_*(X,A;\rho_G) \to \Omega_*(X,A;\rho_{G'})$.

(b) Functors corresponding to different choices of ρ_G are naturally equivalent.

The result follows immediately from theorem 6 and corollary 5 .

With the notations of the theorem, we define $\Omega_*(-;G)$, the p.l. oriented bordism with coefficient group G, by $\Omega_*(X;G) = \Omega_*(X;\rho_G$

8. Corollary For every pair X, A, every $n \geqslant 0$ and every abelian group G, there is a short exact sequence

$$0 \to G \otimes \Omega_n(X,A) \to \Omega_n(X,A,G) \to \mathrm{Tor}(G,\Omega_{n-1}(X,A)) \to$$

which is natural in X, A and in G . $\square$

We are now able to say something about the splitting of the universal-coefficient sequence associated to $\Omega_*(-;G)$. Precisely, since the sequence is natural on the category $\mathcal{Ab}$, Hilton [4], Theorem 3.2, gives us the following

9. Corollary For every abelian group G, the universal-coefficient sequence of $\Omega_n(-;G)$ is pure.

From algebra we deduce:

10. Corollary For every pair X, A; abelian group G and integer $n \geqslant 0$ such that $\mathrm{Tor}(\Omega_{n-1}(X,A),G)$ is a direct sum of cyclic groups, the universal coefficient sequence

$$0 \longrightarrow \Omega_n(X,A) \otimes G \longrightarrow \Omega_n(X,A,G) \longrightarrow \mathrm{Tor}(\Omega_{n-1}(X,A),G) \longrightarrow 0$$

splits.

The class of examples of splitting considered by the previous corollary is quite vast, because it includes the following cases:

(a) any G finitely generated

(b) any G such that its torsion subgroup has finite exponent

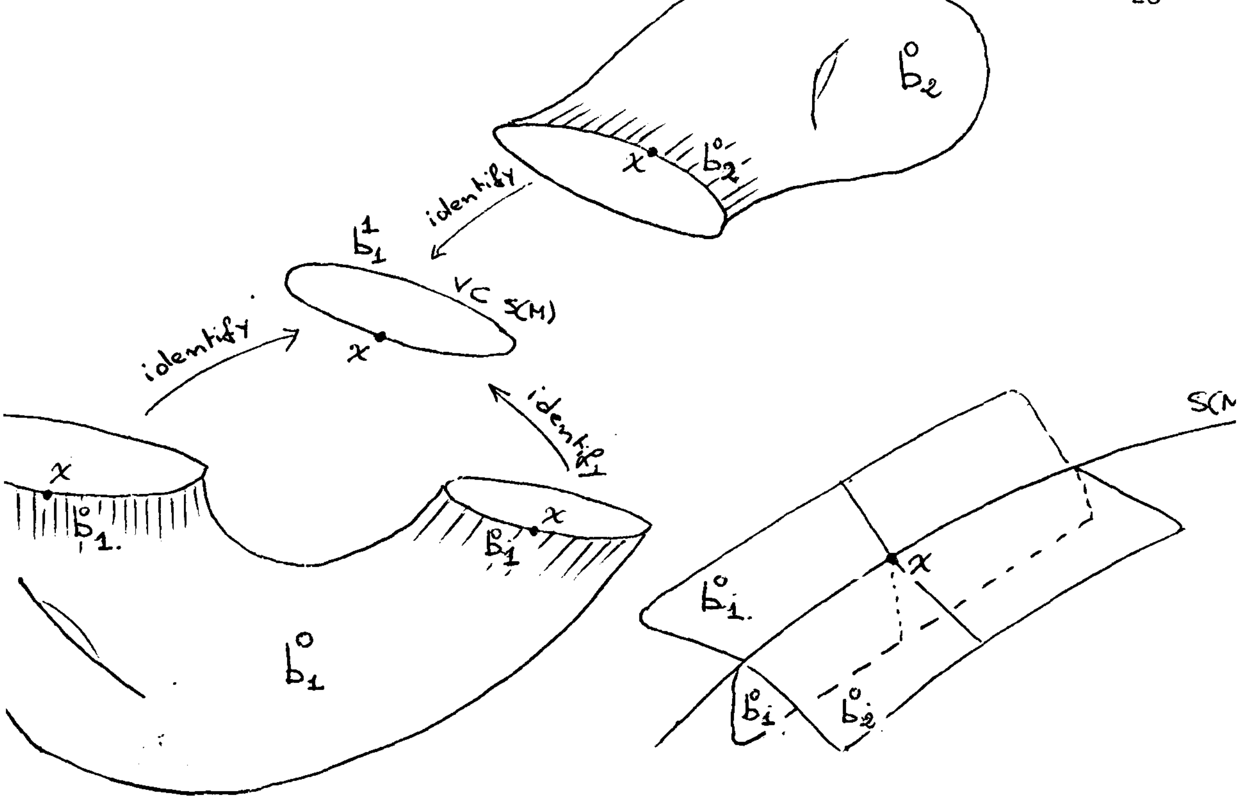
(c) any $\Omega_{n-1}(X,A)$ such that its torsion subgroup has finite exponent.

11. Remark As we have pointed out earlier, the definition of (ς, n)-manifold makes sense in the case of ς being any linked chain complex (not necessarily a resolution) and there is an

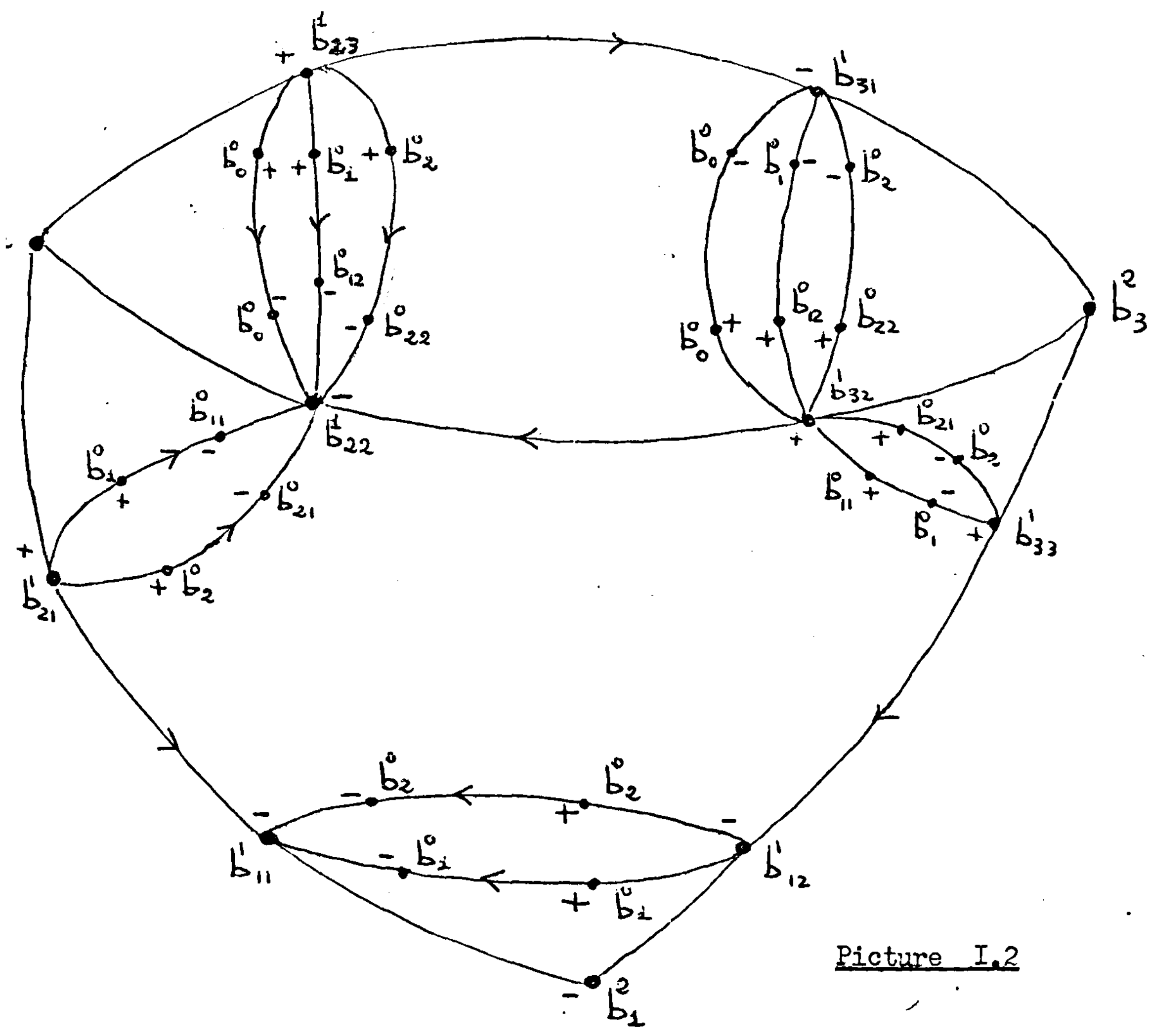
associated bordism theory $\Omega_*(-;\rho)$. Some of the facts about morphisms, that we have established in this section, hold in the case of chain complexes. In particular, if $\tau : \rho \longrightarrow \rho'$ is a linked chain map, ρ is a linked chain complex and ρ' is a linked resolution, then the proof of Prop 1 applies to give an associated morphism $\tau_* : \Omega_*(-;\rho) \longrightarrow \Omega_*(-;\rho')$.

The above treatment of functoriality can be summarized as follows. For every abelian group G, two functors $\mathcal{P} \longrightarrow \mathcal{A}b_*$ have been set up, namely $\Omega_*(X,A;\rho_G)$ and $\tilde{\Omega}_*(X,A;G)$. They have different features: the former is readily seen to be a generalized homology theory; while the latter is natural on the category of abelian groups. Theorem 6 establishes a natural equivalence t_G between the two functors, which proves at the same time that $\Omega_*(X,A;\rho_G)$ is natural on $\mathcal{A}b$ and that $\tilde{\Omega}_*(X,A;G)$ is a homology theory.

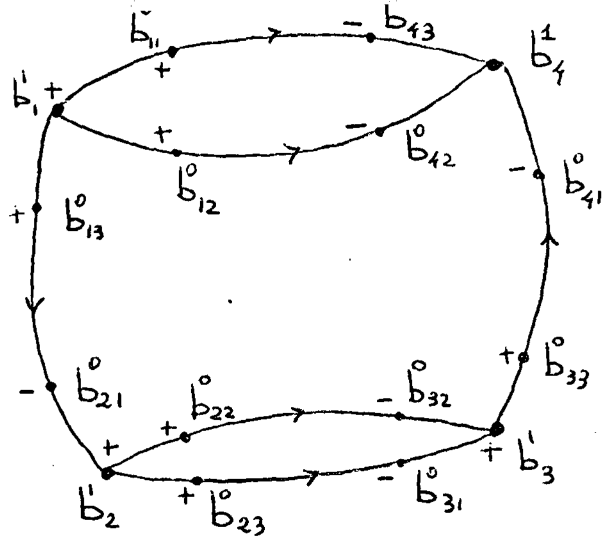
In the following we may use whichever of the equivalent functors $\Omega_*(X,A;\rho_G)$, $\Omega_*(X,A;\rho_i)$, $\tilde{\Omega}$ (X,A;G) is more appropriate to the context ($i = 1,2$; ρ_i = any i-canonical resolution of G) .



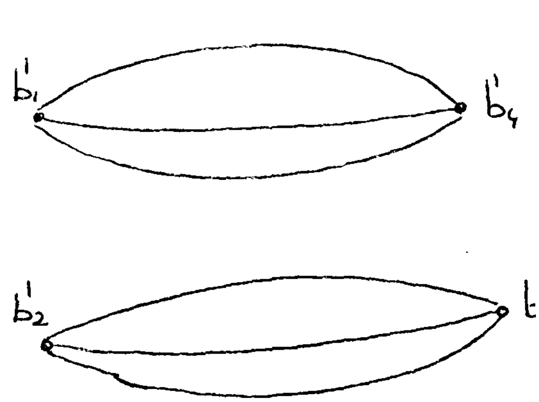
Picture I.1



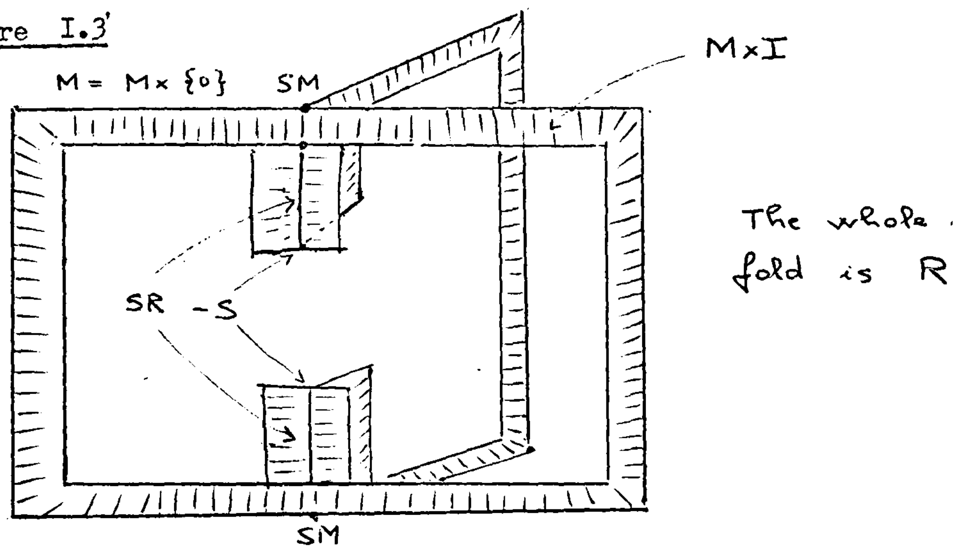
Picture I.2



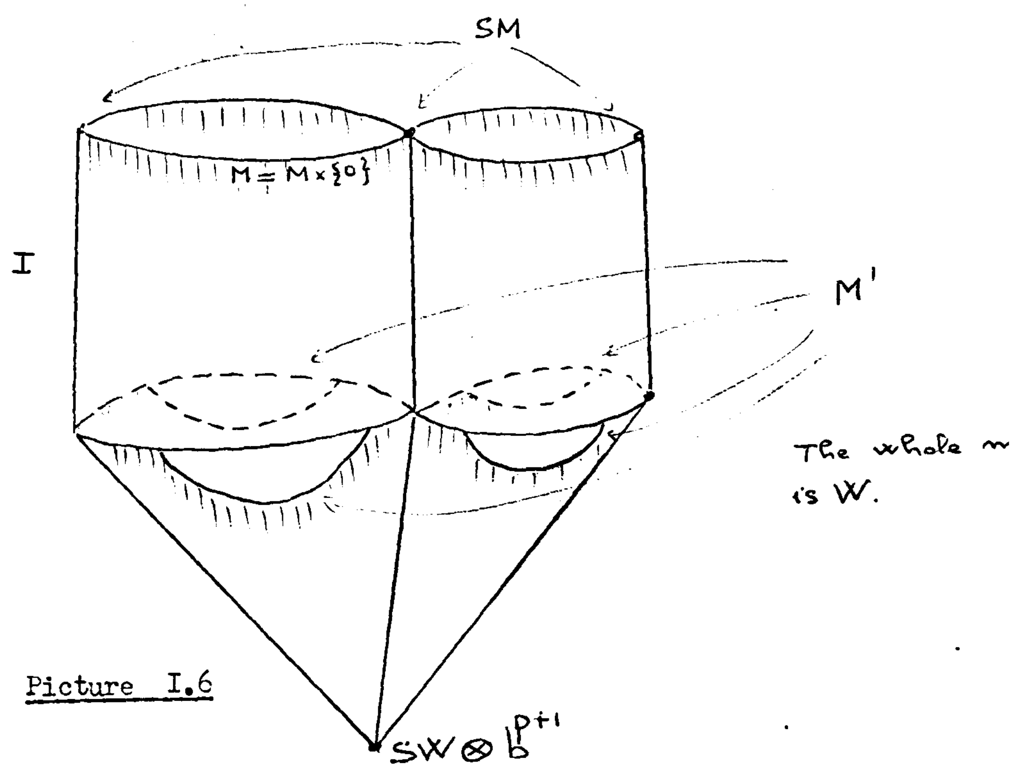
Picture I.3'



Picture I.4



Picture I.5



Picture I.6

CHAPTER II: COBORDISM WITH COEFFICIENTS AND EXTENSIONS

In Section 1 we look at the cartesian product of a (G,i)-manifold and a (G',j)-manifold; the product can be made into a (G $\otimes$ G', i+j)-manifold in a natural way. This fact enables us to define a cross-product homomorphism for bordism with coefficients. The Bockstein sequence, in Section 2, is established by letting its chain-complex analogue act on the labelling of the cycles. In Section 3 we give the definition of cobordism with coefficients using the geometrical approach to cobordism given in [6]. A typical G-cycle is a mock bundle whose blocks are G-manifolds. All desired properties are satisfied (cup, cap-products, Poincaré duality, functoriality on coefficients, etc.). In Section 4 we leave the p.l. category and point out that our method to introduce coefficients works in the range of most geometric co-homology theories. Moreover we use a method of Sullivan's to describe how KO-theory at odd primes can be interpreted geometrically. Section 5 deals with the case of R-modules. If the group G in which we are taking coefficients is also an R-module, the corresponding (graded) bordism group inherits an R-module structure by interpreting an element of R as an endomorphism of G and then applying functoriality.

1. Products

If G, G' are abelian groups, let ρ be a linked resolution

$$0 \longrightarrow F_1 \xrightarrow{\phi} F_o \xrightarrow{\varepsilon} G \longrightarrow 0 \qquad \text{with}$$

$$B^o = G = \{ g_1, g_2, \dots \}$$

$$B^1 = \{ \tau_1, \tau_2, \dots \}$$

and ρ' defined similarly. Then $\rho \otimes \rho'$ is the augmented chain complex $\{ F'', \phi'' \}$

$$0 \longrightarrow F_1 \otimes F_1' \xrightarrow{\phi_2''} F_o \otimes F_1' \oplus F_1 \otimes F_o' \xrightarrow{\phi_1''} F_o \otimes F_o' \xrightarrow{\varepsilon''} G \otimes G' -$$

where
$$\phi_2''(\tau \otimes \tau') = \phi(\tau) \otimes \tau' - \tau \otimes \phi'(\tau')$$

$$\phi_1''(g \otimes \tau') = g \otimes \phi'(\tau')$$

$$\phi_1''(\tau \otimes g') = \phi(\tau) \otimes g'$$

$$\varepsilon'' = \varepsilon \otimes \varepsilon'$$

$\rho \otimes \rho'$ is based by means of B^i, B'^i $(i = 0,1)$; it is structured in dimension one by the structures of ρ and ρ' ; in dimension two we assign to $\tau \otimes \tau' \in B'^2$ the word $w(\tau \otimes \tau') = w(\tau) \otimes \tau' - \tau$ For now we do not fix a cancellation rule.

Let M be a (ρ, i)-manifold with singularities SM; M' a (ρ', j)-manifold with singularities SM'. Form the cartesian product $M \times M'$. It has three intrinsic strata given by

$$(M - SM) \times (M' - SM')$$

$$SM \times (M' - SM') \cup (M - SM) \times SM'$$
$$SM \times SM'$$

On each stratum we can put labels via the tensor product, i.e. if V is a component labelled by x , V' labelled by x', $V \times V'$ is labelled by $x \otimes x'$.

We show that $M \times M'$, with such additional structure, is a $(\rho \otimes \rho', i + j)$-manifold. The first and the second stratum are easily seen to be $\rho \otimes \rho'$-manifolds of the appropriate dimensions and we are going to examine the third stratum. For simplicity assume SM, SM' constantly labelled by τ , τ' respectively, so that $SM \times SM'$ is labelled by $\tau \otimes \tau' \in B''^2$. The basic link of $SM \times SM'$ in $M \times M'$ is topologically the join $L'' = L(\tau, \rho) * L(\tau', \rho')$. L'' with the structure induced by $M \times M'$ is a $(\rho \otimes \rho', 1)$-manifold because $M \times M' - SM \times SM'$ is a $(\rho \otimes \rho')$-manifold and there is a product structure around $SM \times SM'$. The zero-dimensional stratum of L'' is isomorphic to $L(\tau, \rho) \otimes \tau' \cup \tau \otimes - L(\tau', \rho')$ where $\otimes$ is meant to act on the labels. But this represents the word $w(\tau \otimes \tau')$. Thus L'' is a $(\rho \otimes \rho', 1)$-manifold in which the zero-dimensional stratum represents $w(\tau \otimes \tau')$ and the 1-dimensional stratum is a union of disks. Therefore L'' gives a unique cancellatio rule to be assigned to $\tau \otimes \tau'$ in order to have $L'' = L(\tau \otimes \tau', \rho$

$M \times M'$ is then a $(\rho \otimes \rho', i + j)$-manifold (Picture II.1) . We can now define a homomorphism

$$X_{\rho, \rho'}: \Omega_*(-; \rho) \otimes \Omega_*(-; \rho') \longrightarrow \Omega_*(-; \rho \otimes \rho')$$

by

$$X_{\rho, \rho'}([M]_\rho \otimes [M']_{\rho'}) = [M \times M']_{\rho \otimes \rho'}$$

$X_{\rho, \rho'}$ is of degree zero.

Let ρ_3 be a 3-canonical resolution of $G \otimes G'$. Then, by the proof of I 33, there exists a canonical lifting $\widetilde{\mathrm{id}}: \varrho \otimes \varrho' \longrightarrow \varrho_3$ of $\mathrm{id}: G \otimes G' \longrightarrow G \otimes G'$. Therefore we can define a <u>cross-product</u> homomorphism:

$$X_{G,G'} : \Omega_*(-;G) \otimes \Omega_*(-;G') \longrightarrow \Omega_*(-;G \otimes G')$$

by the composition

$$\begin{tikzcd} \Omega_*(-;\varrho) \otimes \Omega_*(-;\varrho') \arrow[r, dashed, "X_{G,G'}"] \arrow[d, "X_{\varrho,\varrho'}"] & \Omega_*(-;\varrho_{G\otimes G'}) \\ \Omega_*(-;\varrho \otimes \varrho') \arrow[r, "\widetilde{\mathrm{id}}_*"] & \Omega_*(-;\varrho_3) \arrow[u, "t^{-1}_{1,3}"'] \end{tikzcd}$$

where $\varrho_{G \otimes G'}$ is a linked presentation of $G \otimes G'$; $\widetilde{\mathrm{id}}_*$ the usual relabelling map (as in the proof of I.3.6) .

1. <u>Remark</u> If an abelian group G is also endowed with a multiplication that makes it into a ring, then we have a <u>product homomorphism</u>:

$$\begin{tikzcd} \mu : \Omega_*(-;G) \otimes \Omega_*(-;G) \arrow[rr] \arrow[dr, "X_{G,G}"'] & & \Omega_*(-;G) \\ & \Omega_*(-;G \otimes G) \arrow[ur, "m_*"'] & \end{tikzcd}$$

where $X_{G,G}$ is the cross-product and $m: G \otimes G \longrightarrow G$ is given by: $m(g \otimes g') = gg'$. The homomorphism μ makes $\Omega_*(\text{point}; G)$ into a ring and if X,A is a pair, $\Omega_*(X,A;G)$ can be given a structure of graded module over the ring $\Omega_*(\text{point};G)$ in the usual way.

2. The Bockstein Sequence

1. Theorem On the category of short exact sequences of abelian groups

$$0 \longrightarrow G' \xrightarrow{\varphi} G \xrightarrow{\psi} G'' \longrightarrow 0$$

there is a natural connecting homomorphism

$$\beta : \widetilde{\Omega}_*(-;\, G'') \longrightarrow \widetilde{\Omega}_*(-;\, G')$$

of degree -1 and a natural long exact sequence

$$\cdots \xrightarrow{\beta} \widetilde{\Omega}_n(-;\, G') \xrightarrow{\varphi_*} \widetilde{\Omega}_n(-;\, G) \xrightarrow{\psi_*} \widetilde{\Omega}_n(-;\, G'') \xrightarrow{\beta} \cdots$$

Proof For the sake of clarity of exposition we prove the theorem under the assumptions: $G' \subset G$ and $(X,A) = (\text{point}, \emptyset)$.

Realize the exact sequence of abelian groups by the (not necessarily exact) sequence of canonical resolutions and linked maps

$$\begin{tikzcd}
 & \Gamma'_2 \arrow[r, hook] \arrow[d] & \Gamma_2 \arrow[r] \arrow[d] & \Gamma''_2 \arrow[d] & \\
 & \Gamma'_1 \arrow[r, hook] \arrow[d] & \Gamma_1 \arrow[r] \arrow[d] & \Gamma''_1 \arrow[d] & \\
 & \Gamma'_o \arrow[r, hook] \arrow[d] & \Gamma_o \arrow[r] \arrow[d] & \Gamma''_o \arrow[d] & \\
0 \arrow[r] & G' \arrow[r, hook] & G \arrow[r] & G'' \arrow[r] & 0
\end{tikzcd}$$

(1) Definition of β Let M^n be a G"-manifold. Suppose that the singularities of M have only one connected component $V \otimes r''$, with $r'' = g_1'' + \dots + g_t''$ a relation in G". We relabel V by an element of G' as follows. Choose $g_1, \dots, g_t \subset G$ such that $\psi(g_i) = g_i''$ and $g_i = g_j \iff g''_i = g_j''$. Then $\psi(g_1 + \dots + g_t) = g_1'' + \dots + g_t'' = 0$ in G". Therefore by exactness $g' = \sum_i g_i$ is an element of G'. We relabel V by g' and get a (G',n-1)-manifold $V \otimes g'$.

Now suppose $\bar{g}_1, \dots, \bar{g}_t$ is another lifting of $g_1'', \dots, g_t''$ as above, giving a (G',n-1)-manifold $V \otimes \bar{g}'$. We show that $V \otimes g'$ and $V \otimes \bar{g}'$ are bordant as (G',n-1)-manifolds. Relabel M/V by changing g_i'' into $g_i' = g_i - \bar{g}_i \in G''$, $i = 1, \dots, t$. The sum $r' = (g_1' + \dots + g_t') - (g' - \bar{g}')$ is a relation in G'. Therefore take a labelled copy $V \otimes r'$ of V and form the polyhedron $W = (V \otimes r') \times L(r', G') \cup M$, where $L(r', G')$ is the link generated by r' in G' and the union is taken along the common part $V \times \mathrm{cone}(g_1' + \dots + g_t')$ (picture II.2). W is a (G',n)-manifold and providesthe required bordism between $V \otimes g'$ and $V \otimes \bar{g}'$.

If the singular set, S(M), of M has more than one component, the relabelling construction can be performed componentwise and one gets a (G',n-1)-manifold, $\beta(M)$, whose bordism class is independent of the various choices.

Next we show that $\beta(M)$ depends only on the bordism class of M, i.e. if $M \overset{G''}{\sim} \emptyset$, then $\beta(M) \overset{G'}{\sim} \emptyset$. Let W be a (G",n+1)-manifold with $\partial W = M$. If $V \subset W/SW$ is a component labelled by $g'' \in G''$, choose $g \in G$ such that $\psi(g) = g''$ and relabel V by g. Let T^n be a component of the n-stratum of W. The sum in G of the new labels on the sheets coming into T^n is an element g' of G'. We relabel T by g'. Finally, if T^{n-1} is a component of the $(n-1)$-stratum of W and $T_1^n \otimes g_1', \ldots, T_s^n \otimes g_s'$ are the sheets merging into T^{n-1}, then $\tau' = \sum_{i=1}^{s} g_i'$ is a relation in G', because T^{n-1} was originally labelled by a "relation amongst relations" of G''. We label T^{n-1} by τ'. If $\beta(W)$ denotes SW with the relabelling described above, then $\beta(W)$ provides the required G'-bordism $\beta(M) \sim \emptyset$.

Now we are entitled to define

$$\beta : \tilde{\Omega}_n(-;\, G'') \longrightarrow \tilde{\Omega}_{n-1}(-;\, G')$$

by

$$\beta([M]_{G''}) = [\beta(M)]_{G'} \quad .$$

(2) <u>Exactness at $\tilde{\Omega}_n(-;\, G)$</u>

(a) $\psi_* \varphi_* = 0$. If M' is a (G',n)-manifold, then $M'' = \psi\varphi(M')$ is a (o,n)-manifold. Therefore $M'' \sim \emptyset$ by the proof of the universal-coefficient theorem.

(b) $\operatorname{Ker} \psi_* \subset \operatorname{Im} \varphi_*$. Let M^n be a G-manifold and W'' a $(G'',n+1)$-manifold with $\partial W'' = \psi(M)$. We show how to modify W'' in order to get a G-bordism between M^n and a G'-manifold M'^n.

Relabel each component of the $(n+1)$-stratum of W'' by elements of G, obtained from the G''-labels through a lifting $G \underset{\leftarrow}{\xrightarrow{\psi}} G''$ such that

(a) The relabelled n-stratum of $\partial W''$ coincides with the n-stratum of M^n.

(b) If two components are labelled by the same element of G'', the corresponding liftings coincide.

Let V be a component of the n-stratum of W'' and $g_1,\dots,g_v$ the new G-labels around V; $g' = \sum_{i=1}^{v} g_i$ is an element of G'. Attach a new sheet $(V \times I) \otimes g'$ to V iff $g' \neq 0$ and label V by the G-relation $\tau = \sum_i g_i - g'$ (Picture II.3). Now let $V \otimes \tau$, $\bar{V} \otimes \bar{\tau}$, $\bar{\bar{V}} \otimes \bar{\bar{\tau}}, \dots$ be the components of the relabelled n-stratum merging into a component T of the (n-1)-stratum. The corresponding new sheets which have been inserted, namely $(V \times I) \otimes g'$, $(\bar{V} \times I) \otimes \bar{g}'$, $(\bar{\bar{V}} \times I) \times \bar{\bar{g}}', \dots$, are, by construction, such that $\tau' = g' + \bar{g}' + \bar{\bar{g}}' + \dots$ is a relation in G'. Therefore we can glue them to one another along a new n-dimensional sheet $(T \times I) \otimes \tau'$. The resulting polyhedron W provides the required G-bordism.

(3) Exactness at $\widetilde{\Omega}_n(-;\, G'')$

(a) $\beta\psi_* = 0$. Let M^n be a G-manifold. According to the definition of β, we have $\beta\psi(M) = \bigcup_V V \otimes 0$, where V varies over the set of components of S(M). But $V \otimes 0 \sim \emptyset$ by a trivial G'-bordism.

(b) $\operatorname{Ker} \beta \subset \operatorname{Im} \psi_*$. Let M''^n be a G''-manifold. For the sake of simplicity let us assume that the singularities of M'' have only one component $V \otimes \tau''$; $\tau'' = \sum_i g_i''$. Then $\beta(M'') = V \otimes g'$, where $g' = \sum_i g_i$, $\psi g_i = g_i''$. By assumption there is a G'-bordism $W' : \beta(M'') \sim \emptyset$. We construct a G-manifold M^n as follows.

(i) Since $\beta(M'')$ has no singularities we can assume that W' has singularities in codimension one at most, because otherwise we solve the codimension-two stratum as in the proof of the universal-coefficient theorem.

(ii) In M'' we replace each g''_i by g_i and $V \otimes \tau''$ by $V \otimes \tau'$, where $\tau' = g' - \sum_i g_i$.

(iii) We attach W' to the relabelled M'' identifying $\partial W'$ with $V \otimes \tau'$.

It is readily checked that the resulting labelled polyhedron M^n is a (closed) G-manifold such that $\psi(M) \overset{G}{\sim} M''$. (Picture II.4)

(4) <u>Exactness at $\tilde{\Omega}_n(-; G')$</u>

(a) $\varphi_* \beta = 0$. Let M''^{n+1} be a G"-manifold with connected singularities $V \otimes \tau''$, $\tau'' = \sum_i g''_i$. Then $\beta(M'') = V \otimes g'$ where $g' = \sum_i g_i$ and $\psi g_i = g''_i$. We construct a G-bordism $W : \beta(M'') \sim \emptyset$ as follows.

(i) We relabel M by changing each g''_i into g_i and τ'' into $\tau = \sum_i g_i - g'$.

(ii) We attach a new sheet $(V \times I) \otimes g'$ to the relabelled M'' along the singularities $V \otimes \tau$. (Picture II.5)

(b) $\operatorname{Ker} \varphi_* \subset \operatorname{Im} \beta$. Let M'^n be a G'-manifold and $W : M'^n \sim \emptyset$ a G-bordism. Out of W we get a G"-manifold W" of dimension (n+1) as follows.

(i) We remove from W all the strata which are labelled by elements of G' or by relations or by "relations amongst relations".

(ii) We relabel the remaining strata according to the map ψ .

The resulting object $\overline{W}''$ is a $(G'', n+1)$-manifold with singularities in codimension $\leqslant 2$ and it is closed because $\partial W = M'$ has been removed in step (i) .

(iii) We get W'' by solving the codimension-two singularities of $\overline{W}''$ up to bordism.

Now we show that $\beta(W'')$ is G'-bordant to M' . Let Q'' be the bordism $S(\overline{W}'') \sim S(W'')$ used in (iii) , where $S(-)$ stands for 'singularities of $(-)$' . Q'' has at most two sheets; the non-singular one is labelled by relations in G'' and the singular one is labelled by 'relations amongst relations'; $\beta(Q'')$ (constructed as in the proof that the Bockstein is well defined) realizes a G'-bordism between $\beta(\overline{W}'')$ and $\beta(W'')$. Therefore we only need to provide a G'-bordism N' between $\beta(\overline{W}'')$ and the original M' . To this purpose we reconsider the G-bordism W and remove from it all the top dimensional strata which are not labelled by elements of G' . The resulting object W_o is not a G'-manifold in general. We show how to make W_o into the required G'-bordism by inserting new sheets.

(i) Let $V \otimes r$ be a component of the n-dimensional stratum of W , with $r = g'_1 + \dots + g'_p + g_1 + \dots + g_q$ $(g'_i \in G'$; $g_j \in G - G')$. Then $V \otimes g' \subset \beta(\overline{W}'')$ where $g' = g_1 + \dots + g_q$. Therefore we attach a sheet $(V \times I) \otimes g'$ to W_o along V and change the label r into $r' = g'_1 + \dots + g'_p + g'$, which is now a relation in G' .

(ii) Let $V \otimes \tau$, $\bar{V} \times \bar{\tau}$, $\bar{\bar{V}} \otimes \bar{\bar{\tau}}$,... be components merging into a component $T \otimes \tilde{\tau}$ of the (n-1)-stratum, where $\tilde{\tau} = \tau + \bar{\tau} + \bar{\bar{\tau}} + \ldots$ is a relation amongst relations in G. The corresponding new sheets which have been inserted, namely $(V \times I) \otimes g'$, $(\bar{V} \times I) \otimes \bar{g}'$, $(\bar{\bar{V}} \times I) \otimes \bar{\bar{g}}'$,..., are, by construction, such that $\tau' = g' + \bar{g}' + \bar{\bar{g}}' + \ldots$ is a relation in G'. Therefore we can glue them together along the n-dimensional sheet $(T \times I) \otimes \tau'$. The resulting polyhedron provides the required G'-bordism $N' : \beta(\bar{W}'') \sim M'$.

The proof of exactness is now complete and naturality is clear. $\square$

3. Cobordism with coefficients

Once a G-manifold has been defined, the description of oriented p.l. cobordism with coefficients in an abelian group G, written $\Omega^*(-; G)$, is straightforward using the mock-bundle picture of p.l. cobordism given in [6]. More precisely, $\Omega^*(-; G)$ will be the mock-bundle theory, that uses G-mock-bundles instead of the ordinary ones. In the following we give the basic definitions and the reader is continuously referred to [6] for the geometric details.

Let K be an oriented cell complex, i.e. one in which each cell is oriented.

A <u>(G, q)-mock bundle</u> ξ^q/K with base K and total space E consists of a (p.l.) projection $p_\xi : E_\xi \longrightarrow K$ such that:

(a) for each $\sigma \in K$, $p_\xi^{-1}(\sigma)$ is the interior of a $(G, q + \dim\sigma)$-manifold $\overline{\xi(\sigma)}$;

(b) for each $\sigma \in K$, $\overline{\xi(\sigma)} = \bigcup_{\sigma_i \leqslant \sigma} [\sigma_i : \sigma] p_\xi^{-1}(\sigma_i)$, where $[\sigma_i : \sigma]$ is the incidence number and indicates a change of orientation in $p_\xi^{-1}(\sigma_i)$ iff it is equal to -1.

The manifold $\xi(\sigma) = \overline{\xi(\sigma)} - \partial\,\overline{\xi(\sigma)} = p_\xi^{-1}(\sigma)$ is called the <u>block</u> over σ. (G, q)-mock bundles, ξ, η, over K, are <u>isomorphic</u>, written $\xi \cong \eta$, if there is a homeomorphism $h : E_\xi \longrightarrow E_\eta$ such that $h|\xi(\sigma)$ is an isomorphism of G-manifolds between $\xi(\sigma)$ and $\eta(\sigma)$. Suppose given ξ/K and $L \subset K$, then the <u>restriction</u> $\xi|L$ is defined by $E(\xi|L) = p_\xi^{-1}(L)$ and $P(\xi|L) = p_\xi| : E(\xi|L) \longrightarrow L$. If ξ_0 and ξ_1 are (G, q)-mock bundles over K, then ξ_0 is <u>cobordant</u> to ξ_1 if there exists a (G, q)-mock-bundle $\eta/K \times I$ such that $\eta|K \times \{i\} = (-1)^i \xi_i$, $i = 0, 1$ ($-\xi$ is obtained from ξ by reversing the orientations in each block). It is easy to see that cobordism is an equivalence relation, $\sim$, and so we can define $\Omega^q(K, L, G)$ to be the set of cobordism classes of mock bundles empty over L, where the cobordisms are assumed to be empty over $L \times I$. The theory of (G, q)-mock bundles is a straightforward generalization of the corresponding theory of mock-bundles without coefficients, developed in [6]. One deduces that there is a cohomology theory $\{\Omega^*(-; G), \partial_*\}$, defined on polyhedra, which is the Poincare-Lefchetz dual of $\Omega_*(-; G)$. We define $\Omega^q(K,G)$ to be the -<u>qth p.l. cobordism group of K with G-coefficients</u>. While we omit to give full details, we discuss the situations requiring some extra attention in setting up the theory with coefficients and its properties.

1. If ξ^q/K is a (G, q)-mock bundle and $|K|$ is an n-manifold, then $E(\xi$ is a $(G, n + q)$-manifold.

Here we must be sure that, in passing from one cell to another, the bundles involved in the G-stratifications of the blocks do not twist, si in that case we would not get a $(G, n + q)$-manifold when we glue up the bundle. But everything is all right because the blocks have compatible trivializations [illegible], so that, if σ_1, σ_2 are n-cells and τ is an (n-1)-dimensional face of both, $\overline{\xi(\sigma_1)} \cup \overline{\xi(\sigma_2)}$ is necessarily a $(G, n+q)$-manifold or $\overline{\xi(\tau)}$ could not be a G-submanifold of both, up to orientation.

2. If $\varphi : G \otimes G' \longrightarrow G''$ is a pairing, the <u>cup product</u> $\Omega^q(-; G) \otimes \Omega^r(-; G') \longrightarrow \Omega^{q-r}(-; G'')$ and the <u>cap product</u> $\Omega^q(-; G) \otimes \Omega_r(-; G') \longrightarrow \Omega_{q+r}(-; G'')$ can be easily defined by means of the usual pull-back construction, because we have already established the existence of a cross-product in 1.

3. In order to prove Poincaré duality, one needs Cohen's transversality theorem in its full generality, i.e. if $f : J \longrightarrow K$ is simplicial, then $f^{-1}(\tilde{A})$ is collared in $f^{-1}(A^*)$ for each $A \in K$. Here A^* is the 'dual cone of A in K and $A^* = a\tilde{A}$, a = vertex of A^*. From such theorem it follows t if $f : E \longrightarrow K$ is a simplicial map, E is a $(G, n+q)$-manifold and $|K|$ is n-manifold, the inverse image of a dual cell in K cuts the singularities E transversally, so that $f : E \longrightarrow K^*$ is the projection of a (G, q)-mock bundle.

4. A discussion completely analogous to that of I.3 can be carried out in the cohomological case. In particular, $\Omega^*(-; G)$ is a functor on th category of abelian groups and there is a universal-coefficient sequence:

$$. \; 0 \longrightarrow \Omega^q(-) \otimes G \longrightarrow \Omega^q(-;G) \longrightarrow \mathrm{Tor}(\Omega^{q-1}(-),G) \longrightarrow 0$$

which is natural on the category of abelian groups.

4. Extension to other cohomology theories

Let $T_*(-)$ be a bordism theory whose cycles belong to a category of polyedra (T_*-manifolds) defined by specifying:

(a) aclass of admissible links for a T_*-manifold

(b) a reduction of the structural group of the normal block-bundle system of a T_*-manifold.

Then, if $\pi^S_*(-)$ denotes stable homotopy (cycles:framed manifolds), there is a natural tranformation $\theta : \pi^S_*(-) \longrightarrow T_*(-)$ given by regarding a framed manifold as a T_*-manifold. Now let C be a framed circle; if θ (C) bounds as a T_*-manifold, then there is no obstruction to constructing the class of links associated to a resolution, in the same way as we did for p.l. bordism $\Omega_*(-)$.

The method used to introduce coefficients in $\Omega_*(-)$ extends to the theory $T_*(-)$ as follows. Given an abelian group G, one defines $T_*(-;G)$-manifolds using exactly the same definitions as for oriented p.l. bordism, the only modification being that a polyhedron now means a stratified set in which the intrinsic strata, instead of being p.l.manifolds, are T_*-manifolds. $T_*(-;G)$-manifolds are the cycles of a theory, $T_*(-;G)$, for which there exists a universal-coefficient theorem as for p.l. bordism. The dual mock-bundle theory, $T^*(-;G)$, defined in the obvious way by means of mock-bundles whose blocks are $T_*(-;G)$-manifolds, will then provide a geometrical definition of ' $T^{-*}(-)$ with G-coefficients '

Examples

1. T(-)=oriented p.l. (co)bordism (cycles:oriented p.l.manifolds)

2. $T(-)$ = oriented smooth (co)bordism (cycles: oriented smooth manifolds).

3. $T(-)$ = singular (co)-homology (cycles : pseudomanifolds).

4. $T(-)$ = any (co)-bordism theory, whose cycles are polydedra with a prescribed type of stratification. (see Stone [7]).

There is a particularly interesting example, belonging to this last class of theories, which we want to discuss in more detail. Let $\Omega_*(-)$ be 'oriented smooth bordism'. It is well known that all torsion of $\Omega_*(\text{point})$ consists of elements of order 2 and that $\Omega_*(\text{point}) \otimes Q$, Q the rational numbers, is a polynomial algebra. If we consider the theory $\Omega_*(-)_{odd} = \Omega_*(-) \otimes Z[1/2]$, i.e. the localization of $\Omega_*(-)$ at odd primes, then $\Omega_*(-)_{odd}$ is a homology theory equivalent to $\Omega_*(-; Z[1/2])$, since $Z[1/2]$ is a torsion free abelian group. By the previous discussions, $\Omega_*(-; Z[1/2])$ is representable by means of (oriented) smooth $Z[1/2]$-manifolds. Now there is a natural transformation of theories: Sullivan: $S : \Omega_*(-)_{odd} \longrightarrow KO_*(-)_{odd}$ where $KO_*(-)_{odd}$ is homology real K-theory and $S(\text{point})$ is a ring homomorphism. Since $\Omega_*(-)_{odd}$ is representable by means of geometric cycles and $\Omega_*(\text{point})$ is a polynomial algebra, $KO_*(-)_{odd}$ is also representable, by a method due to Sullivan, which we explain here in detail, as the whole treatment of coefficients developed in this paper has been inspired by such method.

We place ourselves in the smooth category. Let M be a $(Z[1/2], q)$-manifold representing the bordism class of a free polynomial generator in $\Omega_*(pt; Z[1/2])$.

An <u>(M, n)-manifold</u> is a compact polyhedral pair W, SW such that:

(a) $W - SW$ is a $(Z[1/2], n)$-manifold.

(b) SW is a (possibly empty) $(Z[\frac{1}{2}], n-q)$-manifold.

(c) a regular neighbourhood of SW in W is of the form SW$\times$ cone (M).

Now let $\Omega_*(-;\ Z[\frac{1}{2}],\ M)$ be the bordism theory constructed using (M, n)-manifolds as cycles.

5. Proposition. There is a long exact sequence:

$$\longrightarrow \Omega_n(-;\ Z[\tfrac{1}{2}]) \xrightarrow{f} \Omega_n(-;\ Z[\tfrac{1}{2}],\ M) \xrightarrow{h} \Omega_{n-q}(-;\ Z[\tfrac{1}{2}]) \longrightarrow$$

$$\xrightarrow[\times M]{g} \Omega_{n-1}(-;\ Z[\tfrac{1}{2}]) \xrightarrow{f} \cdots$$

where the morphisms f, g, h are defined as follows : f is the map that regards a $(Z[\frac{1}{2}], n)$-manifold as an (M, n)-manifold (without singularities); h is the restriction to the singularities SW; g is 'crossing with M'.

Proof hf = 0. A $(Z[\frac{1}{2}], n)$-manifold has no singularities SW.

Ker h $\subset$ Imf : let W, SW be an (M, n)-manifold such that SW bounds a $(Z[\frac{1}{2}], n-q)$-manifold B. Then we form the product B $\times$ cone M and attach it to W $\times$ $\{-1\}$ in W $\times$ I by the identity along $-$(SW $\times$ cone M). The resulting (M, n+1)-manifold provides a bordism between W and a $(Z[\frac{1}{2}], n)$-manifold.

gh = 0. If W, SW is an (M, n)-manifold, then gh(W) is the boundary of the complement of a regular neighbourhood of SW in W.

Ker g $\subset$ Imf. Let V be a $(Z[\frac{1}{2}], n-q)$-manifold; g(V) is V $\times$ M and if g(V) bounds a $(Z[\frac{1}{2}], n)$-manifold V', then we can glue the product V$\times$ cone(M) to V' along g(V) and get an (M, n)-manifold W such that h(W) = V.

fg = 0. If V is as above, then V$\times$ cone(M) provides the required bordism of $fg(V)$ to $\emptyset$.

Ker $f \subset$ Img. If W_o is a $(Z[1/2], n)$-manifold and $f(W_o) = \partial W$, where W is an $(M, n+1)$-manifold, then $g(SW)$ is bordant to W_o by a bordisı of $(Z[1/2], n)$-manifolds.

Exactness is completely proved. $\square$

What we have done so far holds for M an arbitrary smooth $(Z[1/2], q$-manifold. In the following we are going to use the fact that M represeı a polynomial generator. The exact sequence of the previous proposition holds for any space X. When X = point, since $[M]$ is a free generator, g is monomorphism, f is an epimorphism and $\Omega_n(\text{point}, Z[1/2], M) \cong \Omega_n(\text{point}, Z[1/2])/\operatorname{Im}(\check{g})$. Passing to the rings we have:

$$\Omega_*(\text{point};\ Z[1/2], M) = \Omega_*(\text{point}, Z[1/2])/([\check{M}]).$$

i.e. we have killed the ideal generated by $[M]$ in $\Omega_*(\text{point}; Z[1/2])$.

Now the polynomial algebra $\Omega_*(\text{point}, Z[1/2])$ is free over the generı $\{[M_1], [M_2], \ldots, [M_k], \ldots \mid \dim M_k = 4k\} = \sum$, where we can assume $I(M_k)=0$ if $k>1$ and $I(M_1)=1$ (I=inex or signature). Therefo: $\Omega_*(\text{point};\ Z[1/2], M_k)$ is a homology theory such that the corresponding ring $\Omega_*(\text{point};\ Z[1/2], M_k) = \Omega_*(\text{point}, Z[1/2])/([M_k])$ is still a free polynomial algebra over the generators $\{[M_1], \ldots, [\widehat{M_k}], \ldots\}$. This suggests the following inductive definition.

Suppose that in $\Omega_*(-;\ Z[1/2])$ we have already killed the generator $\{[M_2], [M_3], \ldots, [M_{k-1}]\}$, leaving out $[M_1]$, and we want to kill $[M_k]$. The theory $\Omega_*(-; Z[1/2], M_2, \ldots, M_i)$, i.e. '$\Omega_*(-;\ Z[1/2])$ with the set generators $\sum_i = \{[M_2], \ldots, [M_i]\}$ killed' will be written $\Omega_*(-; \sum_i)$. Now we assume $\Omega_*(-; \sum_i)$ defined for $i = 2, \ldots, k-1$ and construct $\Omega_*(-;\ \sum_k)$.

A (Σ_k, n)-manifold is a compact pair, W, SW such that

(a) W-SW is a (Σ_{k-1}, n)-manifold.

(b) SW is a $(\Sigma_{k-1}, n-4k-1)$-manifold. Note that $4k = \dim M_k$.

(c) a regular neighbourhood of SW in W is of the form SW$\times$ cone M_k.

Therefore a (Σ_k, n)-manifold has a trivial stratification obtained by merging various (Σ_{k-1}, n)-manifolds along a new stratum of singularities SW, which cuts all the Σ_{k-1}-type-singularities transversally. A typical link in W is a polyhedron of the form $S * M_{i_1} * \dots * M_{i_j}$, where S is a sphere, each of the M_i's belongs to Σ_k and $\dim S + \dim(M_{i_1}) + \dots + \dim(M_{i_j}) = n - 1$. There is a long exact sequence:

$$\Omega_n(-; \Sigma_{k-1}) \xrightarrow{f} \Omega_n(-; \Sigma_k) \xrightarrow{h} \Omega_{n-4k-1}(-; \Sigma_{k-1}) \longrightarrow$$

$$\xrightarrow[\times M_k]{g} \Omega_{n-1}(-; \Sigma_{k-1}) \longrightarrow$$

which is defined and proved exactly as in Proposition 6. Since $[M_k]$ is a generator in $\Omega_*(\text{point}; \Sigma_{k-1})$ we get

$$\Omega_*(\text{point}, \Sigma_k) = \Omega_*(\text{point}, \Sigma_{k-1})/([M_k]) =$$

$$= \Omega_*(\text{point}, Z[1/2])/([M_2], \dots, [M_k]).$$

So now we have a method for killing the generators $\{[M_2], \dots, [M_k], \dots$ one after the other. At each stage we get a theory in which the remaining generators are still free. If we put $\Omega_*(-; \Sigma) = \Omega_*(-; Z[1/2]$ with all the generators killed except the first', then we have the following theorem.

6. <u>Theorem</u> There is a natural equivalence between $\Omega_*(X; \Sigma)$ and $KO_*(X)_{odd}$.

<u>Proof</u> We write $K_*(X)$ for $KO_*(X)_{odd}$. Then $K_*(X) = \{K_o(X), K_1(X), \ldots, \}$ is a homomology theory periodic of order four and

$$K_* = K_*(\text{point}) = \begin{cases} Z[1/2] & \text{if } * \equiv 0(4) \\ 0 & \text{otherwise} \end{cases}$$

We know the following about the map

$$S : \Omega_*(X) \otimes_Z Z[1/2] \longrightarrow K_*(X)$$

(1) If X = point; $\tilde{S}(\text{point}) : \Omega_* \otimes_Z Z[1/2] \longrightarrow K_*$ is given by $S : V \otimes \frac{1}{2^n} \rightsquigarrow I(V). \frac{1}{2^n}$.

(2) S induces an equivalence of theories

$$\Omega_*(X) \otimes_{\Omega_*} K_* \xrightarrow{S_*} K_*(X)$$

where Ω_* acts on $\Omega_*(X)$ in the usual way (cartesian product) and on K_* by the signature; i.e. let $[V] \in \Omega_v$ and $\frac{a}{2^n} \in K_n$. Then $[V]. \frac{a}{2^n} = \frac{I(V)a}{2^n} \in K_{n+v}$.

Therefore in order to prove the theorem we only need to provide an equivalence

$$T : \Omega_*(X) \otimes_{\Omega_*} K_* \longrightarrow \Omega_*(X; \Sigma) .$$

Let $[W] \otimes \frac{1}{2^i} \in \Omega_W(X) \otimes K_n$ $(n = 4p)$. Define

$$T : [W] \otimes \frac{1}{2^i} \hookrightarrow [\, (\underbrace{M_1 \times \ldots \times M_1}_{p\text{-times}} \times W) \otimes \frac{1}{2^i} \,]$$

where $\otimes$ on the right-hand side stands for 'labelled by' and $[M_1]$ is the first element of Σ_1 (the only one which has not been killed). Extend T by linearity.

We now prove that T preserves the 'tensor relations'

$$T(\, [W] + [W'] \otimes \frac{1}{2^i} \,) = T(\, [W] \otimes \frac{1}{2^i} \,) + T(\, [W'] \otimes \frac{1}{2^i} \,)$$

$$T(\, [W] \otimes \frac{1}{2^{i-1}} \,) = T(\, [W] \otimes \frac{1}{2^i} \,) + T(\, [W] \otimes \frac{1}{2^i} \,)$$

$$T(\, [V].[W] \otimes \frac{1}{2^i} \,) = T(\, [W] \otimes [V].\frac{1}{2^i} \,) \quad .$$

The first is obvious by definition. For the second we use r_i-singularities ($r_i = \frac{1}{2^i} + \frac{1}{2^i} - \frac{1}{2^{i-1}}$) to construct a bordism of $W \otimes \frac{1}{2^i} \cup W \otimes \frac{1}{2^i} \cup W \otimes (\frac{-1}{2^i})$ to $\emptyset$. We now check that the third relation is satisfied. We have, by definition

$$T(\, [V].[W] \otimes \frac{1}{2^i} \,) - T(\, [W] \otimes [V].\frac{1}{2^i} \,) =$$

$$[\, (\underbrace{M_1 \times \ldots \times M_1}_{p\text{-times}} \times V \times W) \otimes \frac{1}{2^i} \,] - I(V)[\, (\underbrace{M_1 \times \ldots \times M_1}_{(p+q)\text{-times}} \times W) \otimes \frac{1}{2^i} \,]$$

$$(\, v = \dim V = 4q)$$

$$= [\, (N \times W) \otimes \frac{1}{2^i} \,] \quad \text{where } N = \underbrace{M_1 \times \ldots \times M_1}_{p\text{-times}} \times V - I(V)(\underbrace{M_1 \times \ldots M_1}_{(p+q)\text{-times}})$$

Note that, by construction, the map $N \times W \longrightarrow X$ is of the form $(N \longrightarrow \text{point}) \times (W \longrightarrow X)$ so that N determines an element

$[N\ ]_\Sigma \in \Omega_*(\text{point}; \Sigma)$. We only need to prove that $[N\ ]_\Sigma = 0$. We look at $I(N)$. Since $I(M_1) = 1$, we have $I(N) = I(V) - I(V) = 0$. The fact that N is a manifold with zero signature implies that it can be expressed as a polynomial in the generators

$\Sigma - [M_1] = \{ [M_2], \ldots [M_k] \ldots \}$. But in the theory $\Omega_*(-; \Sigma)$, the cone over an element of $\Sigma - [M_1]$ is an allowable singularity. Thus it is easy to construct a bordism $N \sim \emptyset$ and we have proved that $T(X)$ is a well defined natural transformation of theories. By construction T(point) is an isomorphism. Therefore T is an equivalence of theories and the theorem is proved. $\square$

5 Bordism with coefficients in an R-module

In order to define coefficients in an R-module we need the following additivity lemma.

1. Lemma If $\alpha, \beta : G \longrightarrow G'$ are abelian-group homomorphisms, then $(\alpha + \beta)_* = \alpha_* + \beta_* : \tilde{\Omega}_*(-; G) \longrightarrow \tilde{\Omega}_*(-; G')$.

Proof Consider the chain map $\tilde{\psi} = \tilde{\alpha} + \tilde{\beta} - \widetilde{(\alpha + \beta)}$ where $\tilde{\alpha}, \tilde{\beta}, \widetilde{\alpha+\beta}$ are the canonical liftings of $\alpha, \beta, \alpha + \beta$. If $[M] \in \tilde{\Omega}_n(-; G)$, put $\tilde{\psi}(M) = \tilde{\alpha}(M) + \tilde{\beta}(M) - \widetilde{(\alpha + \beta)}(M)$. Then $\tilde{\psi}(M)$ is a (G', n)-manifold and we only need to prove that $\tilde{\psi}(M) \sim \emptyset$ in $\tilde{\Omega}_n(-; G')$. But $\tilde{\psi}$ is a lifting of the zero map $0 : G \longrightarrow G'$. Therefore there exists a chain homotopy $D: \tilde{\psi} \simeq 0$.

$$\begin{tikzcd}
\Gamma_2 \arrow[r, "\phi_2"] \arrow[d] & \Gamma_1 \arrow[r, "\phi_1"] \arrow[d, "\widetilde{\psi}_1"] \arrow[dl, "D_1"'] & \Gamma_0 \arrow[r, "\varepsilon"] \arrow[d, "\widetilde{\psi}_0"] \arrow[dl, "D_0"'] & G \arrow[r] \arrow[d, "0"] & 0 \\
\Gamma'_2 \arrow[r, "\phi'_2"] & \Gamma'_1 \arrow[r, "\phi'_1"] & \Gamma'_0 \arrow[r] & G' \arrow[r] & 0
\end{tikzcd}$$

that $\widetilde{\psi}_1 = D_0\phi_1 + \phi'_2 D_1$

$\widetilde{\psi}_0 = \phi'_1 D_0$

definition the singularities of $\widetilde{\psi}(M)$ are given by $\widetilde{\psi}(SM)$.

nsider the induced diagram

.

$$\begin{tikzcd}
\Omega_{n-1}(-) \otimes \Gamma_2 \arrow[r, "1\otimes\phi_2"] & \Omega_{n-1} \otimes \Gamma_1 \arrow[r, "1\otimes\phi_1"] \arrow[d, "1\otimes\widetilde{\psi}_1"] \arrow[dl, "1\otimes D_1"'] & \Omega_{n-1} \otimes \Gamma_0 \arrow[d, "1\otimes\widetilde{\psi}_0"] \arrow[dl, "1\otimes D_0"'] \\
\Omega_{n-1}(-) \otimes \Gamma'_2 \arrow[r, "1\otimes\phi'_2"'] & \Omega_{n-1} \otimes \Gamma'_1 \arrow[r, "1\otimes\phi'_1"'] & \Omega_{n-1} \otimes \Gamma'_0
\end{tikzcd}$$

ere $1\otimes D$ is now a homotopy of $1\otimes\widetilde{\psi}$ to zero. As we know, represents an element in $\Omega_{n-1}(-)\otimes\Gamma_1$. So we have

$$\widetilde{\psi}(SM) = (1\otimes\widetilde{\psi}_1)(SM) = (1\otimes D_0)\circ(1\otimes\phi)(SM) + (1\otimes\phi')\circ(1\otimes D_1)(SM) .$$

t $(1\otimes\phi)(SM) \sim \emptyset$ because $[SM]_{\otimes} = \mathrm{Ker}\,(1\otimes\phi_1)$. Therefore SM) $= (1\otimes\phi')(W')$, where $W' = (1\otimes D_1)(SM)$. By the proof of e universal-coefficient theorem this is sufficient to ensure that e singularities $\widetilde{\psi}(SM)$ of $\widetilde{\psi}(M)$ can be solved by a bordism of ',n)-manifolds. So, using the homotopy D, we have eliminated the ngular stratum of $\widetilde{\psi}(M)$. Let V' be the resulting (G',n)-manifold. V'_0 is a component labelled by $g'_0 \in G'$, then there exists $g_0 \in G$

h that $\widetilde{\Psi}_o(g_o) = g'_o$, because the process of resolving the
gularities in $\psi(M)$ does not change the labelling of the top
ensional stratum. Therefore the element of $\Omega_n(-) \otimes \Gamma'_o$
resented by V' is the image of some $[V] \in \Omega_n(-) \otimes \Gamma_o$
ough $1 \otimes \widetilde{\Psi}_o$. Then again we have $V' = \widetilde{\psi}(V) = (1 \otimes \phi')(1 \otimes D_o)(V)$
that V' may be borded to $\emptyset$ by a $(G', n+1)$-manifold with singularities
en by $(1 \times D_o)(V)$. $\square$

We now turn to the main object of this section, i.e. putting
efficients in an R-module. In the following R will be a
mutative ring with unit.

If G is an R-module, let $\Omega_*(-; G)$ be bordism with
efficients in the underlying abelian group G; $\Omega_*(-; G)$ has a
ural R-module structure. In fact, we must exhibit a ring
orphism $\sigma : R \longrightarrow \mathrm{Hom}_z(\Omega_*(-; G), \Omega_*(-; G))$. The
ve additivity lemma, together with functoriality, tells us that
re is a ring homomorphism

$$\sigma' : \mathrm{Hom}_z(G\ G) \longrightarrow \mathrm{Hom}_z(\Omega_*(-; G), \Omega_*(-; G))$$

ined by $\sigma'(f) = f_*$. Therefore we can define σ by the
position

$$\begin{tikzcd} R \arrow[r, "\sigma"] \arrow[dr, "\sigma''"'] & \mathrm{Hom}_Z(\Omega_*(-;\,G), \Omega_*(-;\,G)) \\ & \mathrm{Hom}_Z(G,G) \arrow[u, "\sigma'"'] \end{tikzcd}$$

σ'' is the R-module structure of G.

The pair $\{\Omega_*(-;\,G), \sigma\}$ is '<u>bordism with coefficients in the R-</u>

<u>e G</u>'. The structure σ will be dropped from the notations.

If $f : G \longrightarrow G'$ is an R-homomorphism, then for every $\tau \in R$ we have

tative diagrams

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\tau"] & G' \arrow[d, "\tau"] \\ G \arrow[r, "f"] & G' \end{tikzcd} \qquad \begin{tikzcd} \Omega_*(-;\,G) \arrow[r, "f_*"] \arrow[d, "\sigma(\tau)"'] \arrow[dr, phantom, "\text{functoriality}"] & \Omega_*(-;\,G') \arrow[d, "\sigma(\tau)"] \\ \Omega_*(-;\,G) \arrow[r, "f_*"] & \Omega_*(-;\,G') \end{tikzcd}$$

: $\Omega_*(-;\,G)$ is a functor on the category of R-modules and

omorphism. From the naturality of the Bockstein sequence for

an groups, it follows that there is a functorial Bockstein sequence

e category of R-modules. Summing up, we have the following:

(a) $\Omega_*(-;\,G)$ is a functor on the category of R-modules

(b) $\Omega_*(-;\,G)$ is additive

(c) For every short exact sequence of R-modules, there is an associated functorial Bockstein sequence.

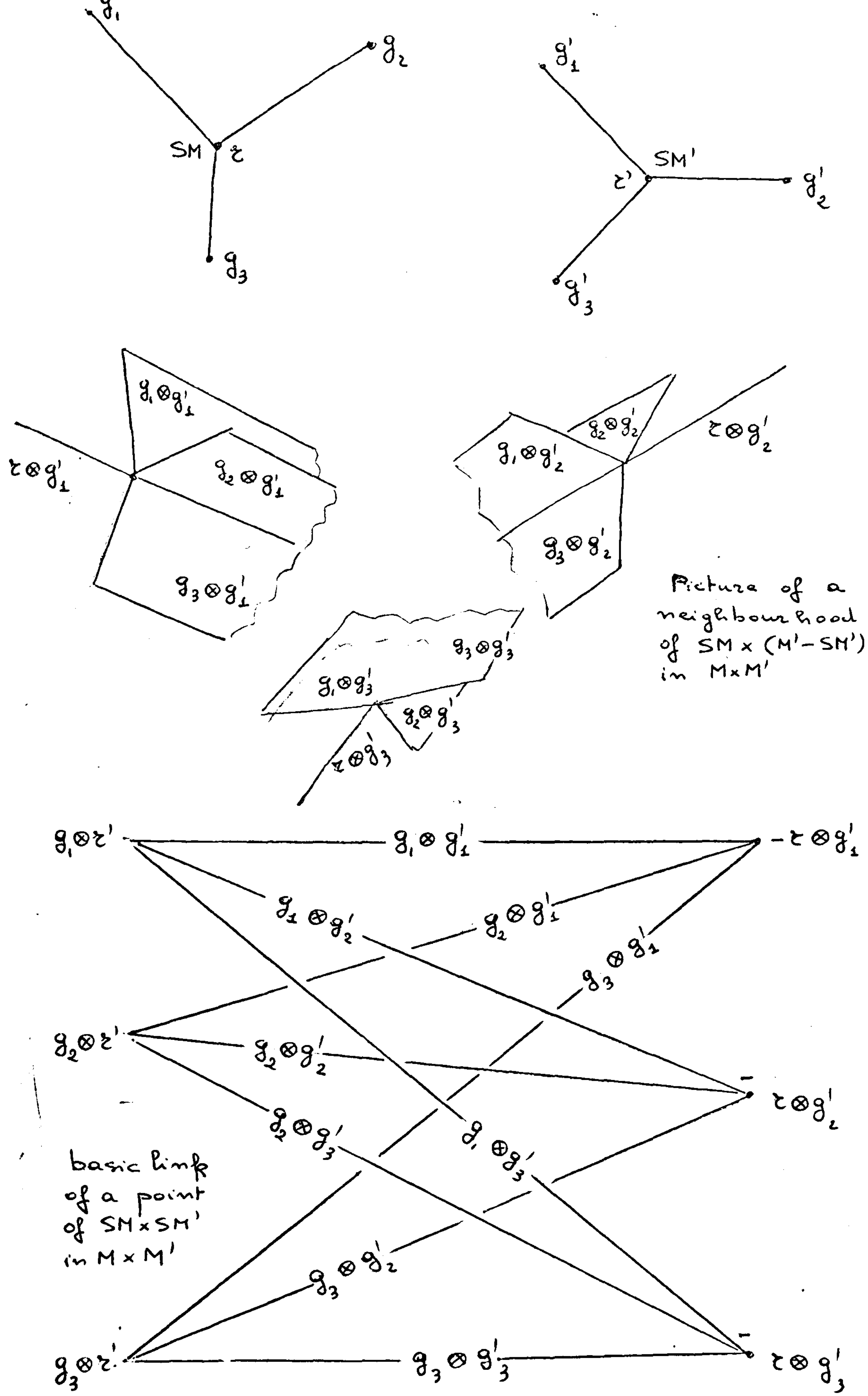
rties (a), (b), (c) form the hypothesis of Dold's Universal-

icient theorem [3].

fore we deduce that there is a spectral sequence running

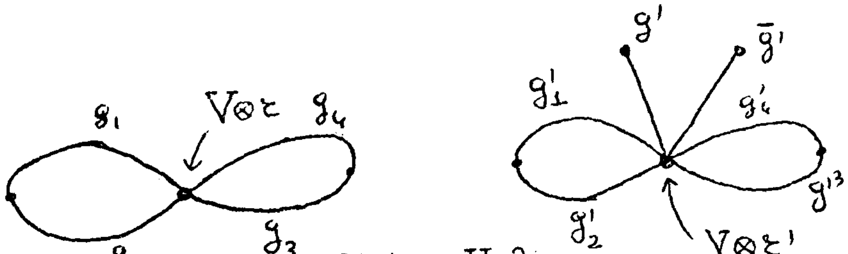
$$E^2_{p,q} = \mathrm{Tor}_p(\Omega_q(-;R), G) \underset{p}{\Longrightarrow} \Omega_*(-;G)$$

This completes the discussion on the case of R-modules as coefficients. the following we shall only deal with abelian groups ; but it is lerstood that everything we say continues to work in the category of ıodules.

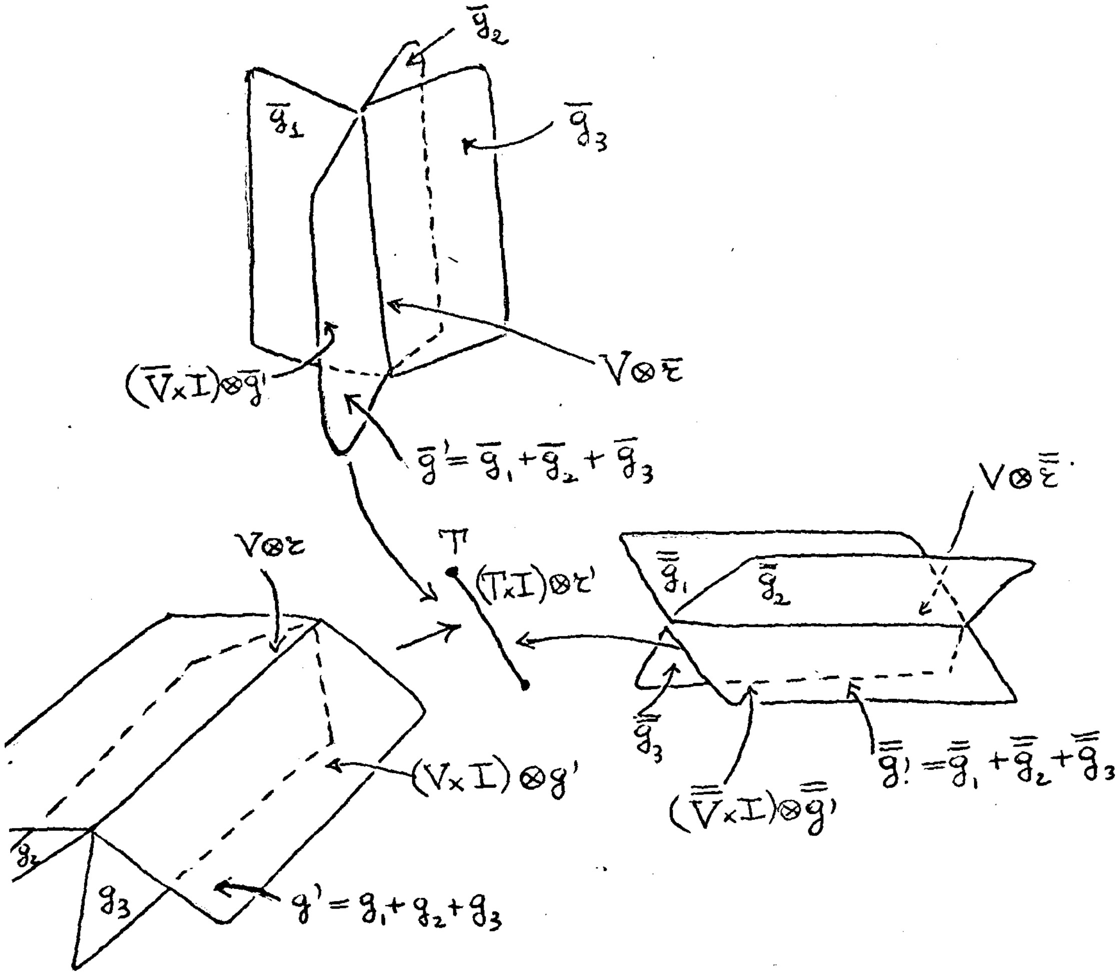


g_1
g_2
g_3
SM
τ
g'_1
g'_2
g'_3
SM'
τ'
$g_1 \otimes g'_1$
$g_2 \otimes g'_1$
$g_3 \otimes g'_1$
$\tau \otimes g'_1$
$g_1 \otimes g'_2$
$g_2 \otimes g'_2$
$g_3 \otimes g'_2$
$\tau \otimes g'_2$
$g_1 \otimes g'_3$
$g_2 \otimes g'_3$
$g_3 \otimes g'_3$
$\tau \otimes g'_3$
Picture of a neighbourhood of SM × (M'−SM') in M×M'
$g_1 \otimes \tau'$
$g_2 \otimes \tau'$
$g_3 \otimes \tau'$
$-\tau \otimes g'_1$
$-\tau \otimes g'_2$
$-\tau \otimes g'_3$
basic link of a point of SM×SM' in M×M'

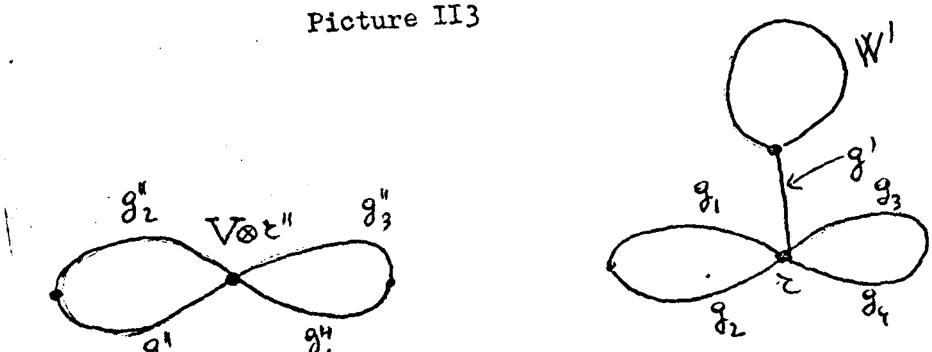
Picture II.1



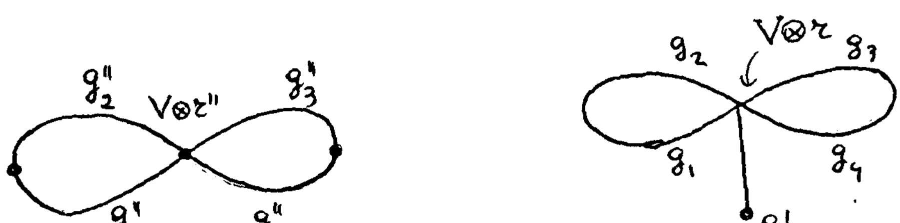
Picture II.2



Picture II3



Picture II.4



Picture II.5

CHAPTER III : COBORDISM WITH LOCAL COEFFICIENTS

In Section 1 we recall the basic facts on stacks and sheaves that will be used in the rest of the chapter. In Section 2 we define the theory of mock bundles with coefficients in a stack. The definition holds in the category of all stacks of abelian groups and it is functorial on this category. The main theorem asserts that, if the stack of coefficients is 'nice', then we have a spectral sequence expressing the relation between simplicial cohomology and cobordism with local coefficients. In Section 3 cobordism with coefficients in a sheaf is described by means of a simplicial analogue of the Cech-procedure. The section ends with an example regarding Poincare duality between bordism and cobordism with coefficients in the local-homology sheaf of a Z_n-manifold.

We remind that in Section 2 we shall be using the same terminology as in the definition of $\widetilde{\Omega}_*(-;G)$ given in I.3 (i.e. a G-manifold has singularities in codimension one at most, while a G-bordism is allowed to have singularities up to codimension two).

1. Stacks

A stack of abelian groups over a cell complex K is a covariant functor $\tau : \underset{\sim}{K} \longrightarrow \mathcal{A}b$, $\underset{\sim}{K}$ denoting the category with objects the cells of K and morphisms the face relations. A homomorphism between stacks, τ/K, τ'/K, is a natural transformation of functors $\varphi : \tau \longrightarrow \tau'$. If τ is a stack over K and $K' \lhd K$, the subdivision of τ over K' is the stack τ'/K' such that $\tau'(\sigma') = \tau(\sigma)$, $\tau'(\sigma_1' < \sigma_2') = \tau(\sigma_1 < \sigma_2)$; $\sigma', \sigma_1', \sigma_2' \in K'$, $\sigma, \sigma_1, \sigma_2 \in K$, $\sigma' \subset \sigma$, $\sigma_1' \subset \sigma_1$, $\sigma_2' \subset \sigma_2$. If τ/K is a stack, $\tau \times I$ will denote the stack over the cell complex $K \times I$, such that : $(\tau \times I)(\sigma \times I) = (\tau \times I)(\sigma) = \tau(\sigma)$ for every $\sigma \in K$; $(\tau \times I)(\sigma_1 \times I < \sigma_2 \times I) = (\tau \times I)(\sigma_1 < \sigma_2) = \tau(\sigma_1 < \sigma_2)$, $\sigma_1, \sigma_2 \in K$. If τ/K is a stack, and $J \subset K$, the restriction $\tau|J$ is defined to be the stack over J, such that $(\tau|J)(\sigma) = \tau(\sigma)$, $\sigma \in J$; $(\tau|J)(\sigma_1 < \sigma_2) = \tau(\sigma_1 < \sigma_2)$, $\sigma_1, \sigma_2 \in J$.

Let X be a polyhedron and F a presheaf of abelian groups over X. If K is a cell complex, $|K| = X$, F induces a stack, F_K/K, by : $F_K(\sigma) = F(st(\sigma, K))$, $F_K(\sigma_1 < \sigma_2) = F(st(\sigma_1, K) \supset st(\sigma_2, K))$, $\sigma, \sigma_1, \sigma_2 \in K$.

We briefly remind the notion of simplicial cohomology with coefficients in a stack and its relation with Check. cohomology. Let τ/K be a stack over the oriented simplicial complex K. A p-cochain, f^p, with coefficients in τ, is a map which assigns to each p-simplex $\sigma^p \in K$, an element of $\tau(\sigma^p)$; p-cochains form an abelian group by coordinate addition, $C^p(K, \tau)$. There is a boundary homomorphism: $\delta^p : C^p(K,\tau) \longrightarrow C^{p+1}(K,\tau)$, given by $\delta^p f^p(\sigma^{p+1}) =$

$$= \sum_{\sigma \leqslant \sigma^{p+1}} [\sigma : \sigma^{p+1}]\, \tau(\sigma < \sigma^{p+1}) f^p(\sigma).$$

$\{C^p(K,\tau), \delta^p\}$ is a cochain complex and its cohomology is by definition $H^p(K,\tau)$.

If F/X is a sheaf and F_K the induced stack over K, let $\mathcal{U}_K$ enote the covering of X formed by the stars of the vertices of K. he nerve of $\mathcal{U}_K$ is known to be equal to K and this fact induces a anonical identification $\left\{ C^p(K,\ F_K), \delta^p \right\} \xrightarrow{\cong} \left\{ \check{C}^p(\mathcal{U}_K, F), \check{\delta} \right\}$ f chain complexes; $\vee$ stands for 'Cech'. Therefore, because the overings by stars are a cofinal system in the system of all overings of X, we get $\lim\limits_{\to K} H^p(K;\ F_K) = \lim\limits_{\to \mathcal{U}} \check{H}^p(\mathcal{U}, F) = \check{H}^p(X;\ F)$.

. <u>Cobordism with coefficients in a stack</u>

Let τ be a stack of abelian groups over an oriented cell omplex K. A <u>(τ, q)-cocycle [cobordism]</u> ξ^q/K consists of a rojection $p_\xi : E \longrightarrow |K|$ such that

(a) for each $\sigma \in K$, $p_\xi^{-1}(\sigma)$ is the interior of a $(\tau(\sigma),\ q + \dim\sigma)$-manifold [bordism], $\overline{\xi(\sigma)}$

(b) for each $\sigma \in K$, $\overline{\xi(\sigma)} = \bigcup_{\sigma_i < \sigma} [\sigma_i : \sigma]\tau(\sigma_i < \sigma)p_\xi^{-1}(\sigma_i)$,

where $\tau(\sigma_i < \sigma)p_\xi^{-1}(\sigma_i)$ is the image of the $\tau(\sigma_i)$-manifold ɔordism] under the morphism $\tau(\sigma_i < \sigma)$; $[\sigma_i : \sigma]$ is the incidence ımber and its effect is a change of orientation iff $[\sigma_i : \sigma] = -1$.

ıe manifold [bordism] $\xi(\sigma) = \overline{\xi(\sigma)} - \partial\overline{\xi(\sigma)}$ is called ıe <u>block</u> over σ ; (τ, q)-cocycles [cobordsims] ξ, η over K :e <u>isomorphic</u>, written $\xi \cong \eta$, if there is a (p.l.) homeomorphism : $E_\xi \longrightarrow E_\eta$ such that $h|\xi(\sigma)$ is an isomorphism of (σ)-manifolds [bordisms] between $\xi(\sigma)$ and $\eta(\sigma)$. Suppose .ven ξ^q/K and $L \subset K$, then the <u>restriction</u> $\xi|L$ is defined in

he obvious way and it is a $(\tau|L, q)$-cocycle [cobordsim] over L.
(τ, q)-cocycle [cobordism] over K,L $(L \subset K)$ is a (τ, q)-cocycle
cobordism] over K, which is empty over L; $-\xi$ is the cocycle
cobordism] obtained from ξ by reversing the orientation in each
lock. If ξ_0 and ξ_1 are (τ, q)-cocycles over K, L, then ξ_0
s <u>cobordant</u> to ξ_1 if there exists a $(\tau \times I, q)$-cobordism
$'/ K \times I, L \times I$ such that $\eta | K \times \{i\} = (-1)^i \xi_i$ $(i = 0,1)$.
obordism is an equivalence relation and we define $\Omega^q(K,L;\tau)$
o be the set of cobordism classes of (τ, q)-cocycles over (K, L);
$\Omega^q(K, L, \tau)$ is an abelian group under the operation of disjoint
nion; we call it the <u>(-q)-th oriented (p.l.) cobordism group</u> of
:, L with coefficients in τ. If $\tau' \lhd \tau$ then there exists an
amalgamation' homomorphism am: $\Omega^q(K', \tau') \longrightarrow \Omega^q(K, \tau)$
.ike in the case of ordinary mock bundles (see [6]). In the proof
f the subdivision theorem for cobordism without coefficients ([6])
ll the geometric constructions are carried out cellwise. Therefore
he proof readily adapts to the present case and we deduce

i. <u>Proposition</u> If $\tau' \lhd \tau$ then am: $\Omega^q(K', \tau) \longrightarrow \Omega^q(K, \tau)$
.s an isomorphism of abelian groups. $\square$

Let $\varphi : \tau_1 \longrightarrow \tau_2$ be a homomorphism of stacks over K.
here is an induced homomorphism $\varphi_* : \Omega^*(K;\tau_1) \longrightarrow \Omega^*(K;\tau_2)$
lefined blockwise by relabelling; more precisely, if ξ/K is a
τ_1-cocycle [cobordism] and $\sigma \in K$, we relabel the block $\xi(\sigma)$ by
neans of $\varphi(\sigma): \tau_1(\sigma) \longrightarrow \tau_2(\sigma)$ (like in I.3). If $\xi'(\sigma)$
is the resulting polyhedron, then the union $\xi' = \bigcup_{\sigma \in K} \xi'(\sigma)$.

is a τ_2-cocycle [cobordism]. All compatibility conditions are ensured by $\varphi : \tau_1 \longrightarrow \tau_2$ being a stack-homomorphism. Therefore if $\mathcal{S}/K$ is the category of stacks over K, we have the following

2. Proposition There is a functor $\Omega^*(K, -) : \mathcal{S} \longrightarrow \mathcal{Ab}_*$ which assigns to each $\tau \in \mathcal{S}$ the abelian group $\Omega^*(K; \tau)$ and to each morphism $\varphi \in \mathcal{S}$ the (graded) abelian-group homomorphism φ_*. $\square$

A linked stack of resolutions over a cell complex K consists of a covariant functor $\rho : \underset{\sim}{K} \longrightarrow \mathcal{E}$ (see I.3 for the definition of $\mathcal{E}$). A linked stack of resolutions ρ is said to be p-canonical if $\rho(\sigma)$ is a p-canonical linked resolution for each $\sigma \in K$ (in the sense of I.3).

If τ/K is a stack of abelian groups and ρ/K is a linked stack of resolutions such that, for each $\sigma \in K$, $\rho(\sigma)$ is a resolution of $\tau(\sigma)$, then we say that τ is represented by ρ. We denote $\mathcal{S}'/K$ the full subcategory of $\mathcal{S}/K$ consisting of all stacks which are representable by p-canonical stacks for each p = 1,2,3. The objects of $\mathcal{S}'$ will also be called nice stacks.

If τ/K is a stack of abelian groups, there is an induced graded stack on K, written $\{\Omega^q_\tau/K\}$ and given by

$$\Omega^q_\tau(\sigma) = \Omega^{-q}(\text{point}; \tau(\sigma))$$
$$\Omega^q_\tau(\sigma_1 < \sigma_2) = \tau(\sigma_1 < \sigma_2)_* .$$

We aim to prove the following

3. <u>Theorem</u> If $\tau \in \mathcal{S}'$, there is a spectral sequence, $E(\tau)$, running

$$E^2_{p,q} = H^p(K; \Omega^q_\tau) \Longrightarrow \Omega^*(K;\tau) .$$

Moreover the spectral sequence is natural on $\mathcal{S}'$.

In order to prove the theorem we need some definitions and lemmas.

Let ϱ be a linked stack over K. A <u>(ϱ,q)-mock bundle</u> ξ^q over K, L consists of a projection $p_\xi : E_\xi \longrightarrow |K|$ such that

(a) for each $\sigma \in K$, $p_\xi^{-1}(\sigma)$ is the interior of a $(\varrho(\sigma), q+\dim\sigma)$-manifold, $\overline{\xi(\sigma)}$;

(b) for each $\sigma \in K$, $\overline{\xi(\sigma)} = \bigcup_{\sigma_i < \sigma} [\sigma_i:\sigma]\varrho(\sigma_i<\sigma)p_\xi^{-1}(\sigma_i)$

(with the usual meaning of the notations)

(c) $p^{-1}(L) = \emptyset$.

Two (ϱ,q)-mock bundles ξ_0, ξ_1 /K,L are <u>cobordant</u> if there exists a (ϱ,q)-mock bundle η/K × I, L × I such that $\eta|K_i = (-1)^i\xi_i$ $i = 0,1$. The <u>(-q th)-cobordism group</u> of K, L with coefficients ϱ, written $\Omega^q(-;\varrho)$, is constructed from (ϱ,q)-mock bundles in the usual fashion.

Thus the main difference between the theory of (τ,q)-cocycles and the theory of (ϱ,q)-mock bundles is that the former allows the cobordisms to have 'more' singularities than the cocycles while the latter is the natural extension of an ordinary mock-bundle theory to the case of local coefficients.

4. <u>Lemma</u> There exists a coboundary homomorphism $\delta^q : \Omega^q(L;\tau) \longrightarrow \Omega^{q-1}(K,L;\tau)$ and a long exact sequence

$$\dots \longrightarrow \Omega^q(K,L;\tau) \xrightarrow{g} \Omega^q(K;\tau) \xrightarrow{f} \Omega^q(L;\tau) \xrightarrow{\delta^q} \dots$$

where g is induced by $(K,L) \longrightarrow (K,\emptyset)$ and f is 'restriction to L' .

<u>Proof</u>

<u>Definition of δ^q</u> It can be roughly described as "pull back onto the boundary of a regular neighbourhood of L in K" . Suppose L full in K . Let $J(1/2)$ be the $\frac{1}{2}$-nd of L in K and $\dot{J}$ its frontier; $\pi : J \longrightarrow L$ the pseudo-radial retraction, $\dot{\pi} = \pi|\dot{J}$. If ξ is a (ρ,q)-mock bundle over L, form the pull back $\dot{\pi}^*(\xi)$, which is a q-mock bundle over $\dot{J}$ and a (q-1)-mock bundle over K, L . If $\xi^*(\sigma)$ is the block of $\dot{\pi}^*(\xi)$ over σ , then $\xi^*(\sigma)$ comes from the block over $\dot{\pi}(\dot{J}(1/2) \cap \sigma)$, denoted $\dot{\pi}(\sigma)$; therefore it has a structure of $\rho(\dot{\pi}(\sigma))$-manifold and it is made into a $\rho(\sigma)$-manifold by means of the stack homo $\rho(\dot{\pi}(\sigma) < \sigma)$. So the resulting object $\dot{\pi}^*(\xi)/K$ is a $(\rho,q-1)$-mock bundle which is empty over L . The assignment $[\xi] \longrightarrow [\dot{\pi}^*(\xi)/K]$ gives a well defined morphism $\delta^q : \Omega^q(L) \longrightarrow \Omega^{q-1}(K,L)$. If L is not full, one first subdivides barycentrically once and then amalgamates.

Exactness is proved geometrically by nice mock-bundle arguments, which are all contained in [6] . The only remark to make is the following.

Suppose $K = \sigma$ and ξ is a (ρ,q)-mock bundle defined on $J = \dot{\sigma} - \sigma_1$, σ_1 face of σ. Then ρ gives morphisms going from the resolutions attached to the simplexes of J to the resolutions attached to σ . Therefore, if ξ is extended over σ by the pull back construction, the resulting block $\xi(\sigma)$ has a natural structure of $\rho(\sigma)$-manifold. $\square$

We remark that the coboundary homomorphism δ^q can be defined also for the theory $\Omega^*(K, L;\tau)$ ($\tau \in \mathcal{S}/K$) exactly in the same way as in the above proof. Therefore there is, for the theory $\Omega^*(K,L;\tau)$, a long sequence analogous to that of Lemma 4. However, although it is immediately checked that the sequence has order two, there is no reason to suppose that it is exact. The argument which is used to prove Lemma 4 fails because in $\Omega^*(K,L;\tau)$ two cobordisms having the same ends cannot be glued together to give a cocycle.

Given a linked stack of resolutions ρ/K there is an associated graded stack $\Omega^q_\rho \in \mathcal{S}$ defined as in the case of abelian-group stacks. We then have the following

5. Lemma If ρ is a linked stack of resolutions there exists a spectral sequence $E(\rho)$ running

$$H^p(K;\Omega^q_\rho) \Longrightarrow \Omega^*(K;\rho) .$$

Proof By lemma 4, for each $p = 0,1,\dots$ we have the exact "p-sequence"

$$\dots \to \Omega^{-p-q+1}(K_p,K_{p-1};\rho) \xrightarrow{g} \Omega^{-p-q+1}(K_p;\rho) \xrightarrow{f} \Omega^{-p-q+1}(K_{p-1};\rho)$$

$$\xrightarrow{\delta} \Omega^{-p-q}(K_p,K_{p-1};\rho) \to \dots$$

where K_p is the p-skeleton of K. The theory of exact couples yields a spectral sequence $E(\varrho)$ in which the chain complex $\left\{ E^1_{p,q};\ d^1_{p,q}: q \text{ fixed} \right\}$ is isomorphic to $\left\{ C^p(K;\Omega^q_\varrho), \delta \right\}$ defined in Section 1. More precisely an isomorphism $h: E^1_{p,q} \longrightarrow C^p(K;\Omega^q_\varrho)$ is given as follows.

$E^1_{p,q} = \Omega^{-p-q}(K_p, K_{p-1}; \varrho)$. Let $[\xi] \in \Omega^{-p-q}(K_p, K_{p-1}; \varrho)$. Then ξ has a block $\xi(\sigma)$ for each p-simplex $\sigma \in K_p$ and $\xi(\sigma)$ is a closed $(\varrho(\sigma), -q)$-manifold, because ξ is empty over K_{p-1}. Therefore $\xi(\sigma)$ determines an element $[\xi(\sigma)]_\varrho \in \Omega^q_\varrho(\sigma)$. We associate to $[\xi]$ a p-cochain $f^p_{[\xi]}$ with coefficients Ω^q_ϱ by setting

$$f^p_{[\xi]}(\sigma) = [\xi(\sigma)]_\varrho \quad \text{for each p-simplex } \sigma \in K .$$

It is readily checked that the correspondence $[\xi] \longrightarrow f^p_{[\xi]}$ defines the isomorphism h.

Then E^∞ is the bigraded module associated to the filtration of $\Omega^*(K;\varrho)$ defined by

$$F^s\Omega^*(K;\varrho) = \operatorname{Ker}\left[\Omega^*(K;\varrho) \longrightarrow \Omega^*(K_{p-1};\varrho)\right] .$$

The lemma follows. $\square$

<u>Proof of Theorem 3</u> For each i = 1,2,3, let ϱ_i be an i-canonical linked stack representing a given $\tau \in \mathcal{S}'$.

There is a commutative diagram of degree-zero homomorphisms:

$$\begin{tikzcd}
\Omega^*(K;\varrho_1) \arrow[rr, "t_{1,2}"] \arrow[dr, "t_{1,3}"'] & & \Omega^*(K;\varrho_2) \arrow[dl, "t_{2,3}"] \\
& \Omega^*(K;\varrho_3) &
\end{tikzcd}$$

in which $t_{i,j}$ is the 'relabelling' map on cocycles. It is easy to see that $t_{i,j}$ commutes with the coboundary operation and therefore there is an induced commutative diagram of spectral-sequence homomorphisms

$$\begin{tikzcd} E(\rho_1) \arrow[rr, "t^*_{1,2}"] \arrow[dr, "t^*_{1,3}"'] & & E(\rho_2) \arrow[dl, "t^*_{2,3}"] \\ & E(\rho_3) & \end{tikzcd}$$

On the E^2-term $t^*_{i,j}$ is the homomorphism $H^p(K;\Omega^q_{\rho_i}) \xrightarrow{t^*_{i,j}} H^p(K;\Omega^q_{\rho_j})$ induced by the change of coefficients $\Omega^q_{\rho_i} \xrightarrow{\text{relabel}} \Omega^q_{\rho_j}$ which we know to be an isomorphism from the theory of coefficients in the constant case.

Then, by the usual spectral-sequence argument, $t^*_{i,j}$ is an isomorphism for all (i,j): $1 \leqslant i < j \leqslant 3$. Now there is a homomorphism $\alpha : \Omega^*(K;\tau) \longrightarrow \Omega^*(K;\rho_3)$ given by the relabelling map on the cocycles. In order to prove the theorem we only need to show that α is an isomorphism.

(1) <u>α is an epimorphism</u> There is a commutative diagram

$$\begin{tikzcd} \Omega^q(K;\rho_1) \arrow[rr, "t_{1,3}"] \arrow[dr, "\theta"'] & & \Omega^q(K;\rho_3) \\ & {}^q(K;\tau) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

where θ is also a relabelling map. Therefore, since $t_{1,3}$ is epi, so is α .

(2) α is a monomorphism Let $[\xi] \in \operatorname{Ker} \alpha$. It follows from the definitions that ξ/K is also a (ρ_2,q)-mock bundle. Therefore it determines a class $[\xi]_{\rho_2}$ which is mapped to zero by $t_{2,3}$. Because $t_{2,3}$ is a monomorphism there exists a (ρ_2,q)-cobordism $\eta : \xi \sim \emptyset$. But η is also a (τ,q)-cobordism. Therefore α is mono.

The theorem follows. $\square$

6. Corollary If τ/K is a constant stack (i.e. $\tau(\sigma) = G$ for each $\sigma \in K$), then $\Omega^*(K;\tau)$ coincides with $\Omega^*(K;G)$ as defined in II 3 .

Proof Since $\Omega^*(K;G)$ is a cohomology theory, there is a spectral sequence E(G) running

$$H^p(K,\Omega^*(\text{point},G)) \Longrightarrow \Omega^*(K;G)$$

There is a natural transformation of cohomology theories

$$t : \Omega^*(K;G) \longrightarrow \Omega^*(K;\tau)$$

given by relabelling. The induced homomorphism of spectral sequences $E(G) \longrightarrow E(\tau)$ is an isomorphism on the E^2-term because

$$\Omega^*_\tau = \Omega^{-*}(\text{point},G) .$$

Therefore the corollary follows from the 'mapping theorem between spectral sequences'. $\square$

7. Corollary If $\tau \in \mathcal{S}'$ and we take the theory $\Omega^*(K)$ to be $H^*(K,Z)$ (= simplicial cohomology with Z-coefficients), then $\Omega^*(K,\tau)$ coincides with the usual definition of simplicial cohomology with coefficients in τ (see Section 1).

Proof $H^*(K;Z)$ is the mock-bundle theory whose blocks are oriented pseudomanifolds. Since a G-pseudomanifold of dimension greater than zero is bordant to $\emptyset$ for every group G, we see that the E^1-term of the spectral sequence $E(\tau)$ reduces to the cochain complex $C^*(K,\tau)$ considered in Section 1 and the spectral sequence collapses. $\square$

3. Cobordism with coefficients in a (pre)-sheaf

We are now ready to give a notion of cobordism with coefficients in a presheaf, using an analogue of the Check procedure. Let X be a polyhedron and F/X a presheaf of abelian groups. If K is a cell-complex triangulation of X, then, by the previous section, we have a graded group $\{\Omega^q(K, F_K)\}$, where F_K is the induced stack on K. Suppose $K' \lhd K$. We define a homomorphism $\alpha_{K,K'} : \Omega^q(K, F_K) \longrightarrow \Omega^q(K', F_{K'})$ as follows: let ξ^q/K be an (F_K,q)-mock bundle. Subdivide ξ over K' and get ξ''/K' such that $\xi''(\sigma')$ is an $F_K(\sigma)$-manifold for $\sigma' \subset \sigma$. The inclusion $st(\sigma', K') \subset st(\sigma, K)$ gives a restriction homomorphism $F_{\sigma,\sigma'} : F_K(\sigma) \longrightarrow F_{K'}(\sigma')$ and we make $\xi''(\sigma')$ into an $F_{K'}(\sigma')$-manifold, $\xi'(\sigma')$, by applying such homomorphism, i.e. $\xi'(\sigma') = F_{\sigma,\sigma'}(\xi''(\sigma'))$. When all the blocks of ξ'' have been relabelled by means of the restriction homomorphisms, one takes care of the orientations in the blocks, so that the incidence numbers ar preserved. The functoriality of the presheaf F ensures that the final object is an $(F_{K'},q)$-mock bundle ξ'/K', called an <u>F-subdivision of ξ over K'</u>. Two F-subdivisions ξ', $\bar{\xi}'$ of ξ over K' are cobordant by the same construction applied to $K \times I$ and $K' \times I$ modulo the ends. Therefore we have a well defined homomorphism

$$\alpha_{K,K'} : \Omega^q(K, F_K) \longrightarrow \Omega^q(K', F_{K'})$$

$$\alpha_{K,K'} : [\xi] \longrightarrow [\xi']$$

The collection of groups and homomorphisms $\{\Omega^q(K, F_K), \alpha_{K,K'}\}$, indexed by the directed sst of all triangulations of X, is a direct system and we define the -<u>qth (p.l.) cobordism group of X with coefficients in F</u> to be the graded group:

$$\Omega^q(X,F) = \lim_{\to K} \{\Omega^q(K,F_K), \alpha_{K,K'}\} .$$

We now remind that a presheaf F/X is <u>locally constant</u> if there is an open covering $\mathcal{U} = \{U\}$ of X, such that, if $U \in \mathcal{U}$ and $x \in U$ then $F(U) = \varinjlim \{F(V)\}$ where V varies over the open neighbourhoods of x. It then follows that, if K is a sufficiently small triangulation of X, i.e. the associated star-covering is a refinement of $\mathcal{U}$, the cohomology of K with coefficients in the stack F_K coincides with the Cech-cohology $\check{H}(X; F_K)$ and the cobordism of K with coefficients in F_K coincides with $\Omega^*(X; F)$ (subdivision theorem 2.1) . We call such an F_K/K a limit stack for F/X . We say that a locally constant sheaf F/X is nice if it has a limit stack which is nice in the sense of Section 2 .

As in the case of stacks, to each presheaf F/X there is associated a graded presheaf $\{\Omega^q_F/X\}$ defined by $\Omega^q_F(U) = \Omega^{-q}(\text{point}; F(U))$, $\Omega^q_F(U \supset V) = F(U \supset V)_*$; U,V open sets of X. If F/X is locally constant, then Ω^q_F/X is also locally constant.

We have the following analogue of theorem 2.3 .

1. Theorem On the category of nice sheaves there is a natural spectral sequence running

$$\check{H}^p(X;\Omega^q_F) \Longrightarrow \Omega^*(X;F) \quad .$$

The proof is the same as that of 2.3, using limit stacks.

A direct consequence of the 'mapping theorem between spectral sequences' is the following comparison theorem

2. Proposition Let F/X be a nice presheaf. If $h : T^*(-) \longrightarrow S^*(-)$ is a natural transformation of cohomology theories, inducing an isomorphism of the corresponding graded sheaves

$$h_F : T^*_F \longrightarrow S^*_F$$

then h induces an isomorphism in local coefficients:

$$h^*(X) : T^*(X;F) \longrightarrow S^*(X;F) \quad . \quad \square$$

Finally we would like to point out that, should we disregard the general setting in which we have defined coefficients and place ourselves in the particular case of a linked stack of resolutions ρ over a complex K, our method of putting coefficients ρ may still provide some extra information about $|K|$, as it happens in the following example regarding Poincarè Duality.

3. <u>Example</u> Let P, SP be a (ρ, m)-manifold, where ρ is the resolution $0 \longrightarrow Z \xrightarrow{3} Z \longrightarrow Z_3 \longrightarrow 0$ and let $\mathcal{L}_m$ be the sheaf of local m-dimensional homology over P. $\mathcal{L}_m$ is determined by any local-homology stack τ/K, $|K| = P$. We have $\tau(\sigma) = F(x_\sigma)$ = free abelian group over one element x_σ, if $\sigma \in K - SK$; $\tau(\sigma') = F(X_1, X_2)$ if $\sigma' \in SK$. The morphisms $\tau(\sigma' < \sigma)$ are so defined:

(a) if $\sigma', \sigma \in K - SK$, $\tau(\sigma' < \sigma)$ is the isomorphism mapping generator into generator

(b) if $\sigma' \in SK$, $\sigma \in K$, then $\tau(\sigma' < \sigma)$ is one of the following assignments depending on which sheet σ belongs to

$$(*) \quad \begin{cases} X_1 \longrightarrow X_\sigma \\ X_2 \longrightarrow X_\sigma \end{cases} \qquad \begin{cases} X_1 \longrightarrow -X_\sigma \\ X_2 \longrightarrow 0 \end{cases} \qquad \begin{cases} X_1 \longrightarrow 0 \\ X_2 \longrightarrow -X_\sigma \end{cases}$$

(c) if $\sigma', \sigma \in SK$, then $\tau(\sigma' < \sigma)$ is the canonical isomorphism.

In order to compute $\Omega^*(K, \tau)$ we can choose, the stack of resolutions given by

$$0 \longrightarrow F(X_\sigma) \xrightarrow{id} F(X_\sigma) \longrightarrow 0 \quad , \qquad \sigma \in K - SK$$

$$0 \longrightarrow F(X_1^\sigma, X_2^\sigma) \longrightarrow F(X_1^\sigma, X_2^\sigma) \longrightarrow 0 \quad , \quad \sigma \in SK$$

The stack homomorphisms are not quite based because zero is not a basis element. However it can be proved easily that components labelled by zero can be removed (or 'coborded off' by a trivial cobordism) without affecting the cobordism class of the cocycle under consideration. Therefore a(τ, q)-mock bundle ξ^q/K will look like Picture III.1 In a block over a simplex $\sigma \in SK$ there is a manifold without singularities, each component of which is labelled by either X_1^σ or X_2^σ and the morphisms of the stack ensure that only two blocks merge into it, so that no singularities are created in the glueing process. Moreover the '-' signs in the stack homomorphisms put the orientations right in passing from one sheet to another across SK. Hence the total space E_ξ is a p.l. oriented manifold and the operation of 'glueing and disregarding the labels' gives a homomorphism $\psi : \Omega^q(K;\tau) \longrightarrow \Omega_{m+q}(K)$. We now prove that ψ is an epimorphism. Let $W^{m+q} \xrightarrow{f} K$ be a bordism class and let K^* be the dual complex of K. Then Cohen's generalised transversality theorom already stated in II.2.3 tells us that

(a) out of $N = SK \times \text{cone}\ (3\ \text{pts}) \subset K$, f is the projection of a (τ, q)-mock bundle

(b) $f^{-1}(\partial N)$ is collared in $f^{-1}(N)$.

Let N_1, N_2, N_3 be the three sheets around SK and suppose N_3 is such that, if $\sigma \in N_3$, the first of the assignments (*) holds. We are going to vary f by a homotopy over N. Take collars $f^{-1}(\partial N_j) \times I \subset N$, j = 1, 2 and stretch them along the cells of K* to get a new map $f' : W \longrightarrow K^*$, which coincides with f outside $N - \partial N$ and is such that $f^{-1}(\partial N_j) \times 1 \subset \subset f'^{-1}(SK)$ j = 1, 2. By a similar collaring argument get also $f'' : W \longrightarrow K$ such that

(a) $f'' = f$ outside $N - \partial N$.

(b) $f''^{-1}(SK) = f^{-1}(\partial N_j) \times 1 \qquad j = 1, 2$

Now label the blocks over SK by X_j (j = 1, 2) and the blocks over SK - K by the unique generator of the appropriate simplex. Then $f'' : W \longrightarrow K^*$ with such labelling is the projection of a (τ, q)-mock bundle ξ''/K such that $\psi[\xi''] = [W]$ (see Picture III 2). This shows that ψ is an epimorphism. The injectivity of ψ follows from the same arguments applied to $K \times I$ modulo $K \times \{0\}$, $K \times \{1\}$. For simplicity we have discussed the Z_3 case, but everything generalises to Z_n in the obvious way. Thus we have established the following generalization of Poincare' duality theorem.

If P is an m-dimensional polyhedron such that:

(a) its singularities, SP, are in codimension one at most

(b) a regular neighbourhood of SP in P is of the form SP × cone (n-points)

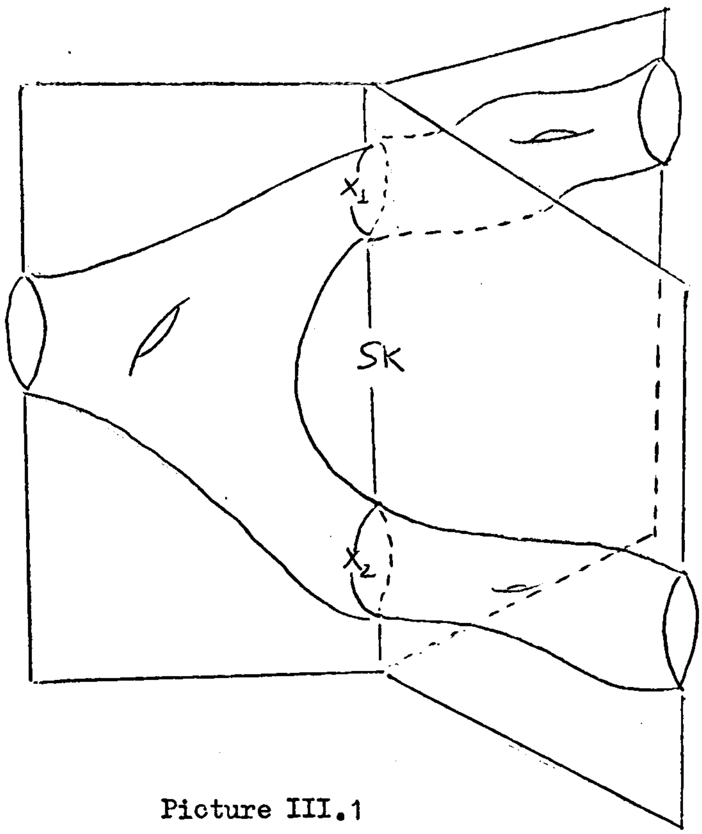
(c) each stratum of P is oriented,

there is a duality isomorphism

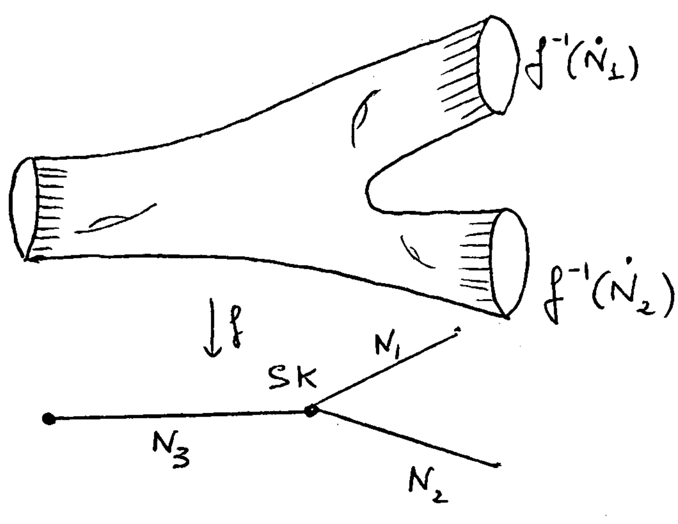
$$\psi : \Omega^q(P, \alpha_m) \longrightarrow \Omega_{m+q}(P). \quad \square$$

In particular, if $SP = \emptyset$, P is an oriented m-manifold and Ψ is the usual Poincare duality. If $n = 2$, P is a p.l. manifold which has singularities of the orientation type. Then Ψ is [illegible] duality with coefficients in a bundle of groups. However, the proposition does not deal with all unorientable manifold, because hypothesis (b) excludes, for instance, the two-dimensional projective space.

The above proposition is particularly stimulating, because it suggests the possibility that generalizations of Poincare duality similar to Zeeman's spectral sequence $H^p(X, \mathcal{L}_q) \Longrightarrow H_*(X)$ (see [9]), might hold for any geometric (co)-homology theory, like, for instance, p.l. cobordism.



Picture III.1



Picture III.2

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